CIV102-STRUCTURES and MATERIALS

Topic: An Introduction to Engineering Design

1) Contact Information

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Note: Multiple right answers.

2) Structural Engineering

The art and science of designing and making or constructing structures with economy and elegance so they can safely resist the forces to which they may be subjected.

Key Terms:

- Art: Best for structures to look good
- Science: Best for structures to stand up
- Making: Best for structures to be feasible
- Economy: Best for structures to be affordable
- Elegance: Best for structures to have style
- Safely: Best for structures to be safe against outside forces

Engineering (from latin) { ingeniare → to create ; ingenium → cleverness }

3) 3 Principles of Engineering

1. $F=ma$ → If somethings not moving, the forces on it must be balanced
2. You Cannot Push on a Rope → Assume you can push on a rope, then constrain that you can't
3. To find the answer, you must know the answer

4) Significant Figures

Use 3.5 significant figures:

If the first non-zero digit is 1, use 4 significant figures
If the first non-zero digit is not 1, use 3 significant figures

5) Engineering Notation

24.3×10^{6}

↳ The exponent is a multiple of 3

CIV102 - STRUCTURES and MATERIALS

Topic: 1D/2D Equilibrium

1) 1D Equilibrium

$F = ma$ Statics → No Movement
Dynamics → Movement

Big changes from 1D → 2D
Not many from 2D → 3D

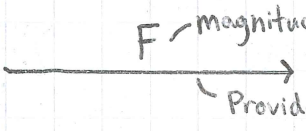
What's the axis?

Forces called actions and reactions
Important to think what is positive and negative

Force: A force is a push or a pull that causes a body to move or change shape
(push or pull: has a directionality; move: acceleration)

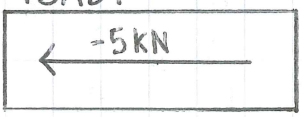
Because a force has a direction and a magnitude, therefore it is a vector.

Force represented by a vector: $\vec{F}$ ← Skip the vector in the notation



Examples:

¡BAD!



Units:
- KN (Kilonewton)
- N (newton)
- lb (pound)
- kip (kilopound)

Body: Something with mass

Examples:

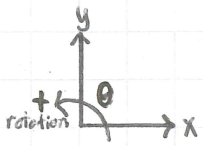
$F = ma$ because statics (No motion)
$\underline{a = 0}$

$\therefore F = \text{Sum of Forces} = (+F_1 - F_2) = 0$

Big Rule:
$\sum F_x = 0$ → Equation of Equilibrium

2) 2D Equilibrium

Axis:

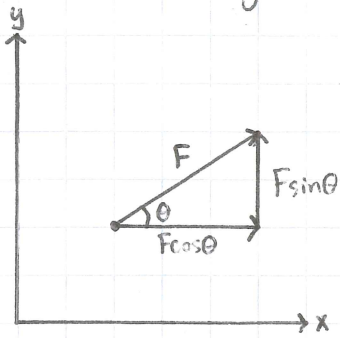


Possible Motions

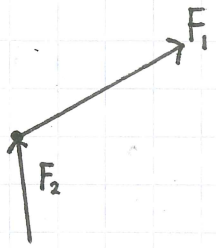
1D → 1
2D → 3
3D → 6

All torques are moments, but not all moments are torques.

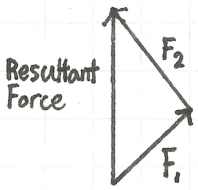
a) Translation Degrees of Freedom (DoF)



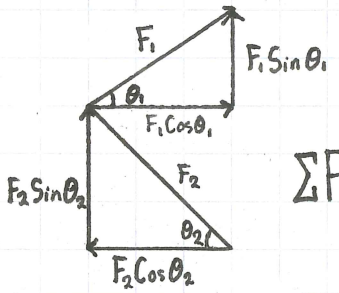
$$\boxed{\begin{aligned}\Sigma F_x &= 0\\ \Sigma F_y &= 0\end{aligned}}$$

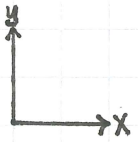


i) Force Polygon



ii) Force Components





$$\Sigma F_x = (F_1 \cos\theta_1 - F_2 \cos\theta_2)$$

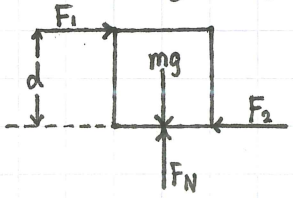
↑ Points to the left

$$\Sigma F_y = (F_1 \sin\theta_1 + F_2 \sin\theta_2)$$

Normal Force: Perpendicular to the Surface
Friction Force = μ Normal Force

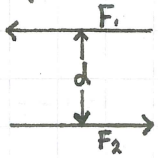
b) Rotations

Free Body Diagram



$$\Sigma F_y = 0$$
$$F_1 = F_2$$
$$\Sigma F_x = 0$$

Couple: Pair of forces acting on the same body that cancel out

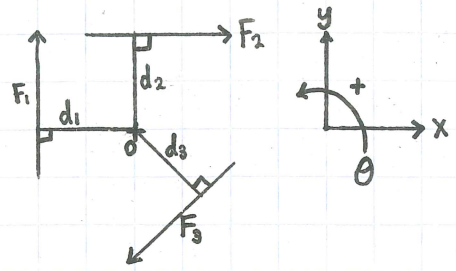


Moment = $F \cdot d$ ← Perpendicular distance between 2 forces

Units: kNm, kilonewton meter

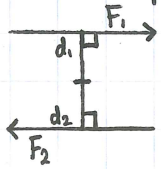
dot product → energy
cross product → moment

More General Case



$$\Sigma M = \underbrace{-F_1 d_1 - F_2 d_2 - F_3 d_3}_{\text{Makes it spin Clockwise}}$$

Check Couples



$$\Sigma M = -F \cdot \frac{d}{2} - F \cdot \frac{d}{2}$$
$$= -F \cdot d$$

Symbol for M

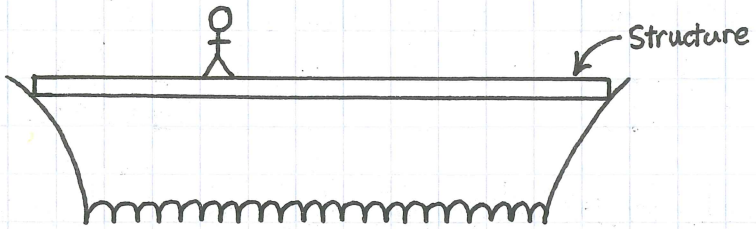
CIV102 - STRUCTURES and MATERIALS

Topic: Building Bridges

1) Structures

An object designed/intended to transmit forces from one place to another.

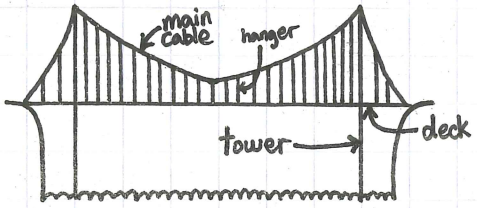
Example:



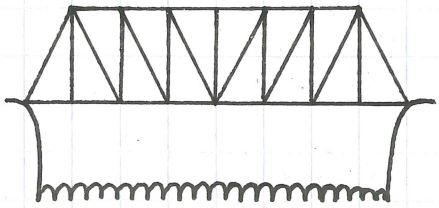
Component: Pieces assembled to make the structure.

Joints/Nodes: Locations where the components connect.
- Must be sufficiently strong and deformable

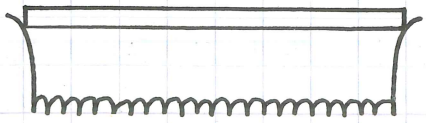
3 Primary Kinds of Bridges



Suspension Bridge



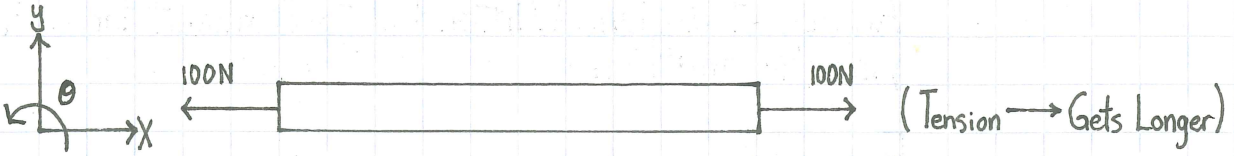
Truss Bridge



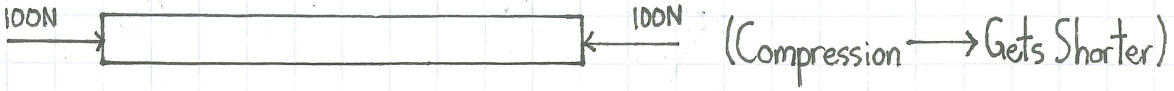
Beam Bridge

2) Tension and Compression

Push-in = Compression } With respect to the FBD
Pull Out = Tension }

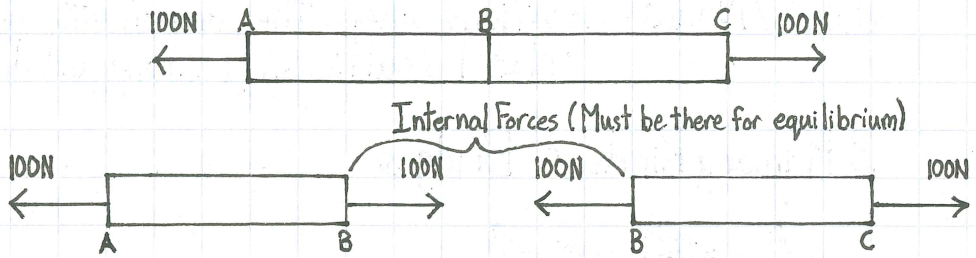


Resisting tension is an internal force. The act of pulling is an external force.



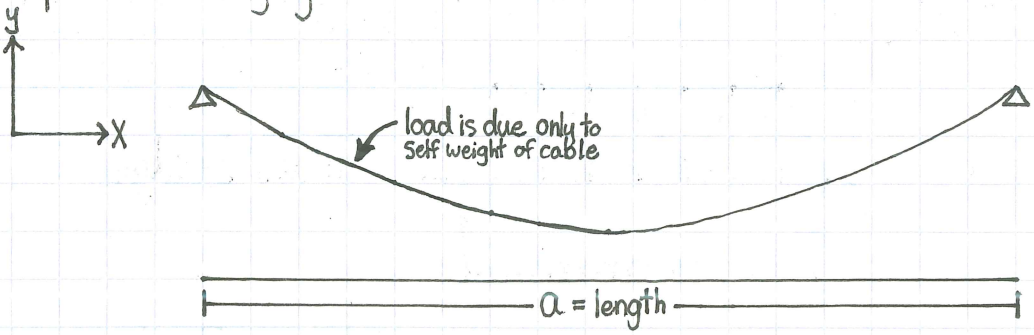
Axial Loads: Tension and compression are axial loads. Axial means they are aligned with the axis of the member.

What is the internal force?



$\Sigma F_x = 0$ $\Sigma F_x = 0$

3) Shape of a Hanging Cable



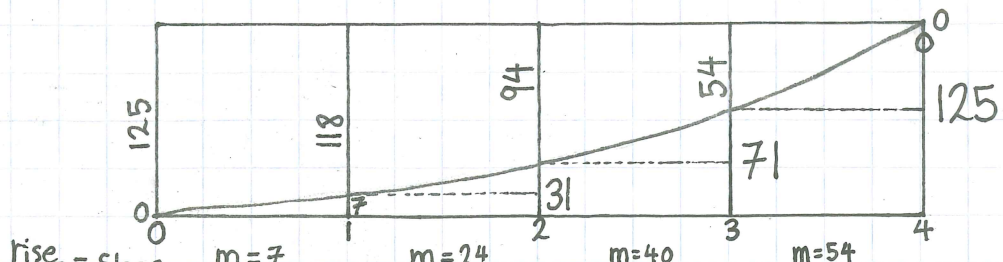
The shape of this curve is called a Catenery (pattern if load is only coming from the cable). The curve can be described by the following equations:

↳ load is distributed compared to the arc length

$y = a\cosh\left(\frac{x}{a}\right) + b$ } Same curve. Different mathematical notation
$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) + b$ }

In suspension bridges, load is distributed uniformly compared to X-axis not the arc length. ∴ Shape is different

What is the shape?



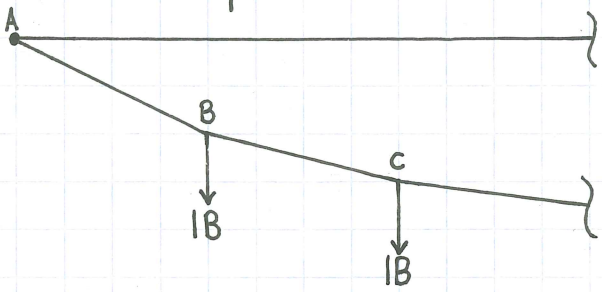
$\frac{\text{rise}}{\text{run}}$ = Slope

m = 7 m = 24 m = 40 m = 54

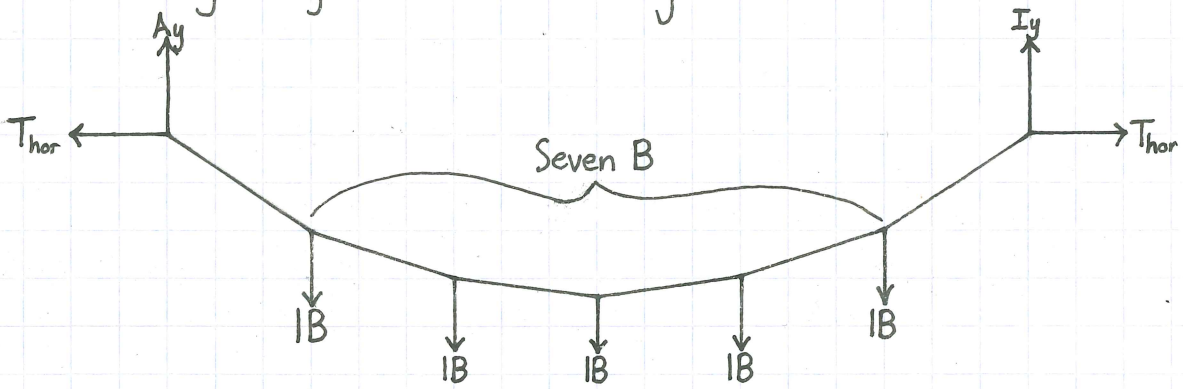
Δm = 17 Δm = 16 Δm = 14

almost the same

Parabolic Shape! Easier than cosh



Free Body Diagram of Whole Thing



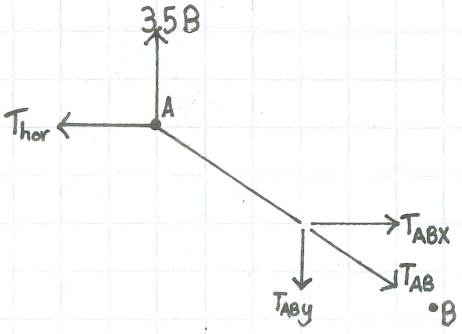
$$\Sigma F_y = 0$$

$$0 = A_y + I_y - 7 \cdot B$$

from symmetry $A_y = I_y$

$$A_y = 3.5B$$

Draw FBD from A → B



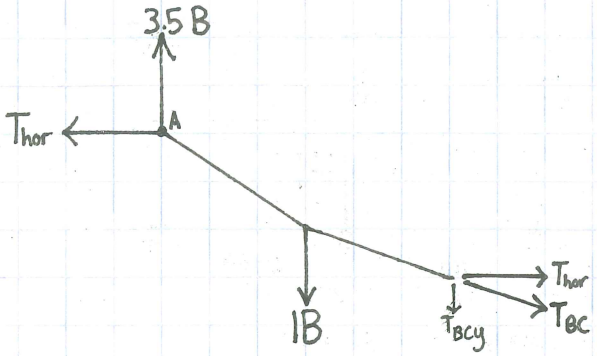
$$\Sigma F_y = 0$$

$$T_{ABy} = 3.5B$$

$$\Sigma F_x = 0$$

$$T_{ABX} = T_{hor}$$

Next One



$\Sigma F_y = 0$

$0 = 3.5B - B - T_{BCy}$

$T_{BCy} = 2.5B$

$\Sigma F_x = 0$

T_{hor} = Constant for all X values

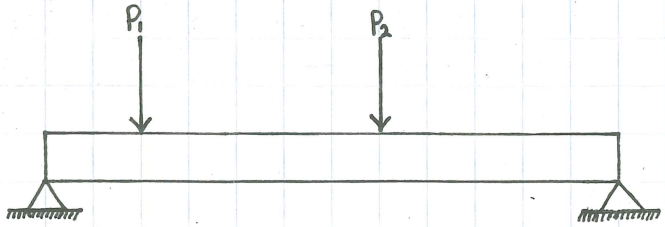
Vertical Component ≠ Constant

CIVI02 - STRUCTURES and MATERIALS

Topic: Big Question - What is the horizontal component of force in a suspension bridge?

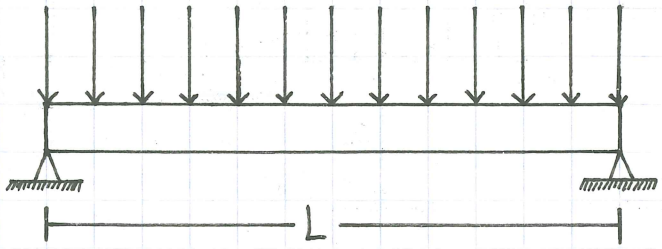
② Distributed Loads:

a) Point Loads



$$\text{Total Load} = \sum P_i$$

b) Uniformly Distributed Load (UDL)



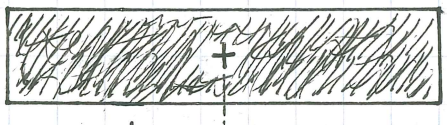
$W = \frac{kN}{m}$

$$\text{Total Load} = W \cdot L = W$$

Moments with UDL

Moment = Total Load × Distance to Centroid of Body Diagram

Loading Diagram

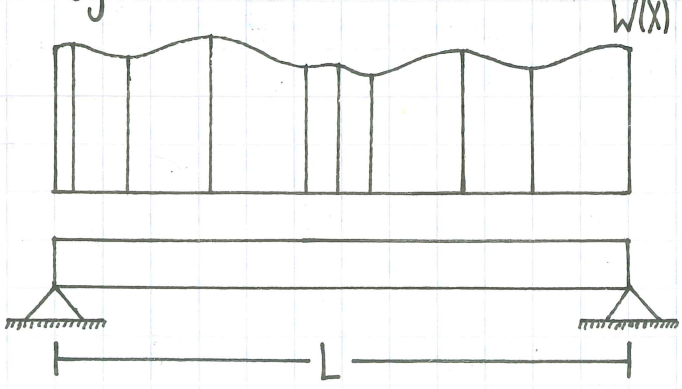


Point of Rotation

$d = \frac{L}{2}$

$$\text{Moment} = \underbrace{(W \cdot L)}_{\text{Total Force}} \cdot \underbrace{\frac{L}{2}}_{\text{Moment Arm or Lever Arm}}$$

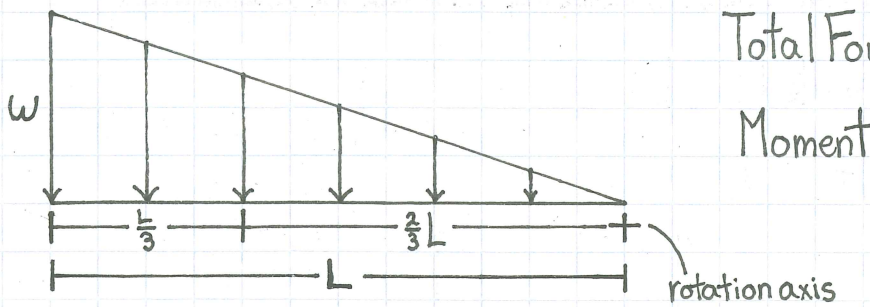
Or by calculus:



$$\text{Total Load} = \int_0^L W(x)\,dx$$

$$\text{Moment} = \int_0^L W(x) \times dx$$

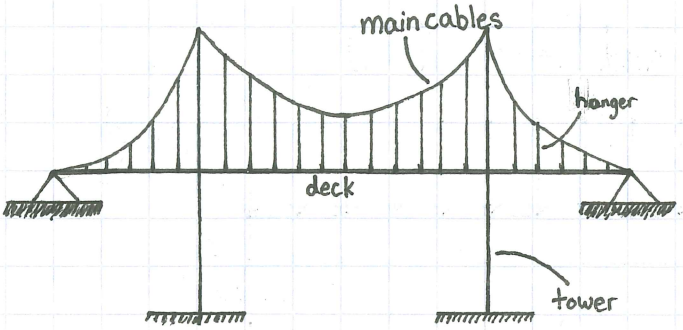
Example 2

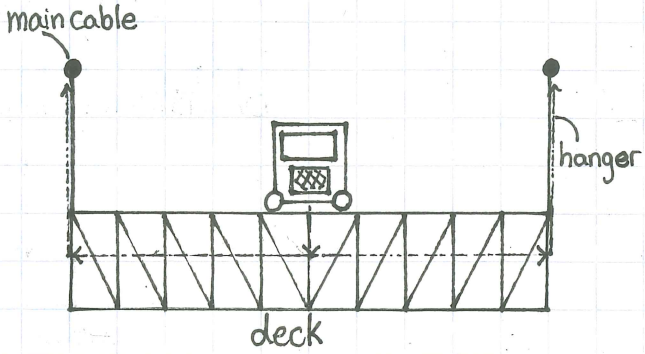


$$\text{Total Force} = \frac{L \cdot w}{2}$$

$$\text{Moment} = +\frac{wL}{2}\cdot\frac{2L}{3} = \frac{wL^2}{3}$$

Suspension Bridges

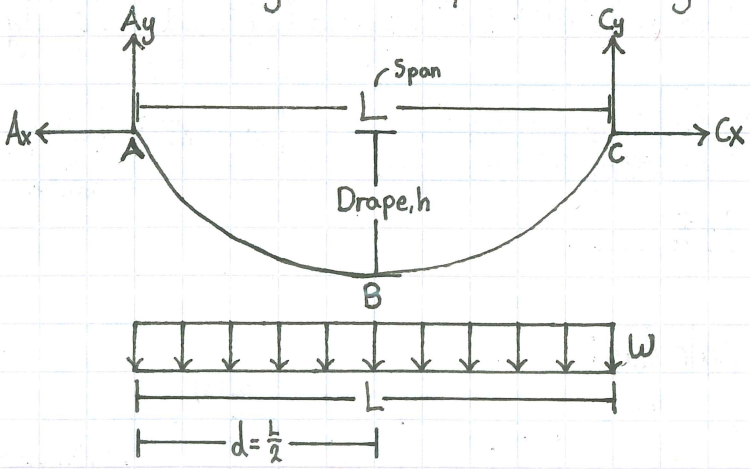




Load Paths

Car (or other load) → Deck → Hanger → Cables → Tower → Ground

③ Structural Analysis of Suspension Bridge



Solve for Ay, Cy

$$\Sigma F_y = 0 = A_y + C_y - w \cdot L$$

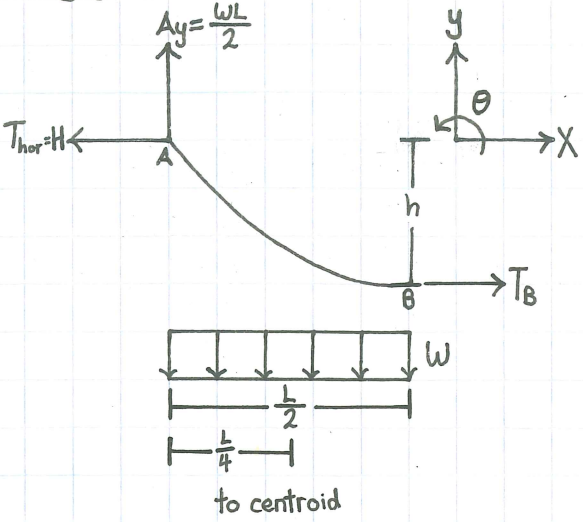
$$\Sigma M_A = 0$$

$$0 = -w \cdot L \cdot \frac{L}{2} + C_y \cdot L$$

$$C_y = \frac{wL}{2}$$

$$\Sigma F_y = 0 \quad A_y = \frac{wL}{2}$$

FBD for A → C



$\Sigma F_x = 0$

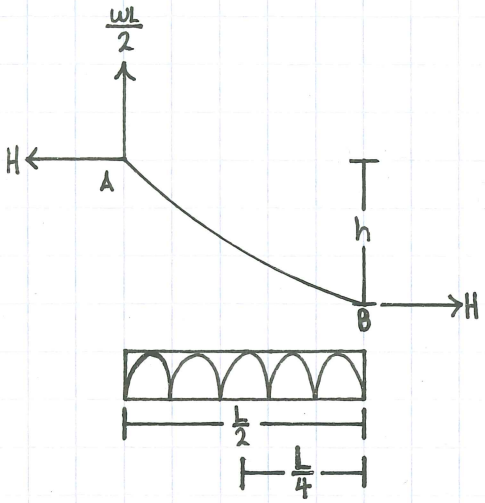
$0 = -T_{hor} + T_B, \; T_{hor} = H$

$0 = -H + T_B, \; T_B = H$

$\Sigma M_A = 0$

$0 = -\frac{wL}{2}\frac{L}{4} + T_B \cdot h \quad (T_B = H)$

$\boxed{H = \frac{wL^2}{8h}}$



$\Sigma M_B = 0$

$0 = -\frac{wL}{2}\frac{L}{2} + Hh + \frac{wL}{2}\frac{L}{4}$

$0 = -\frac{wL^2}{4} + \frac{wL^2}{8} + Hh$

$\frac{wL^2}{8} = Hh$

$\boxed{H = \frac{wL^2}{8h}}$

Golden Gate Bridge

$L = 1280\text{m}$

$h = 143\text{m}$

$w = 370\frac{\text{KN}}{\text{m}}$

$\frac{wL}{2} = 237\text{mN}$

$H = 530\text{mN} \longrightarrow 265\text{mN per Cable}$

Maximum Cable Force

$T_{max} = \sqrt{H^2 + \left(\frac{wL}{2}\right)^2}$

$= 580\text{mN} \longrightarrow 290\text{mN}$ per cable

CIV102 - STRUCTURES and MATERIALS

Topic: Stress, Strain, and Young's Modulus

1) Design Main Cable

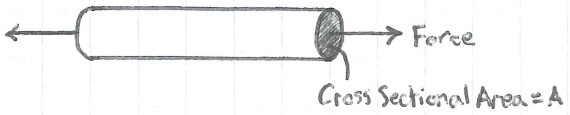
(a) Select a material

(b) Select the size of material needed

Key Considerations:

- Strength → related to forces (kN)
- Deformability → displacement (mm)

2) Consider Only the Material Part



$$\text{Stress} = \sigma = \frac{\text{Force}}{\text{Area}}$$

- Units: $\left[\frac{N}{m^2}\right]$ or [Pa] (Pascal)

CIV102 will use [MPa] (Mega-pascal = 1,000,000 Pa)

- $[MPa] = \left[\frac{MN}{m^2}\right]$

Consistent Set of Units:

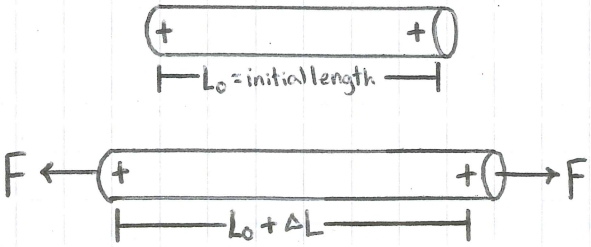
- [N], [mm], [MPa]

$$1[MPa] = \frac{1[MN]}{[m^2]}\left(\frac{1\times10^6[N]}{1[MN]}\right)\left(\frac{1[m]}{1000[mm]}\right)^2$$

$$\Rightarrow 1[MPa] = \frac{1[MN]}{[m^2]} = \frac{1[N]}{[mm^2]}$$

(Other Possible Units: [lb], [inch], [psi])

3) Strain



$$\text{Strain} = \varepsilon = \frac{\Delta L}{L_o} \quad \text{(Unitless)}$$

Often Strains are small, ∴ Call $1\times10^{-3} = \frac{[mm]}{[m]}$

$3\times10^{-3} = 3\frac{[mm]}{[m]}$

Engineering Stress and Strain

- L_o, A = Constants

True Stress and Strain: L_o, A ≠ Constants

Robert Hooke (1678)

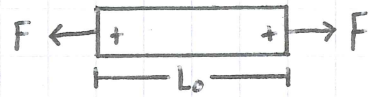
"ceiiinosssttuv"

↳ "ut tensio sic vis"

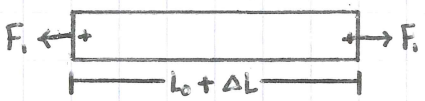
Means: As the extension, so the force.

Hooke's Law

- There is a linear relationship between how hard you pull on something, and how much it gets longer



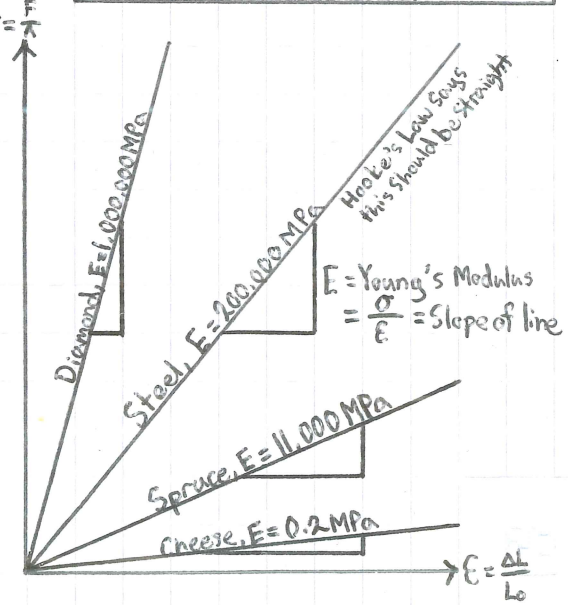
F = 0 (Undeformed Shape)



$F_1 \neq 0$ (Deformed Shape)

Hooke's Law: $F = k(\Delta L)$

- k is the spring constant
- Depends on:
  - Material
  - Length, L_o } Geometry
  - Cross sectional area }



4) Compare Hooke's Law to Hooke's Law!

$F = k(\Delta L)$ Structural Component

$\sigma = E \cdot \varepsilon$ Material Level

$\sigma = \frac{F}{A} \quad \varepsilon = \frac{\Delta L}{L_o}$

$\frac{F}{A} = E \cdot \frac{\Delta L}{L_o}$

$F = \left(\frac{EA}{L_o}\right) \Delta L$

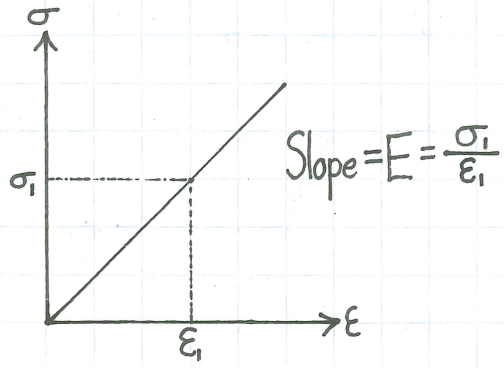
$\therefore \boxed{k = \frac{EA}{L_o}}$

CIV102 - STRUCTURES and MATERIALS

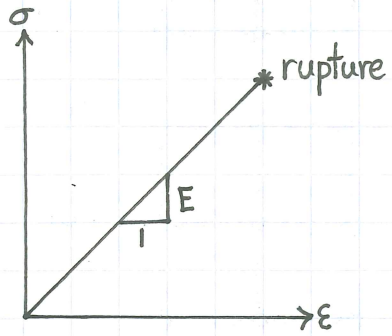
Topic: Mild Steel, Strain Energy, and Material Properties Table

1) $\sigma = \frac{F}{A}$ and $\varepsilon = \frac{\Delta L}{L_o}$

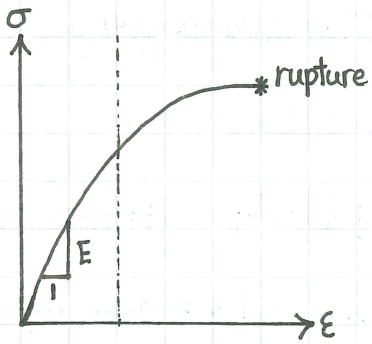
Hooke's Law



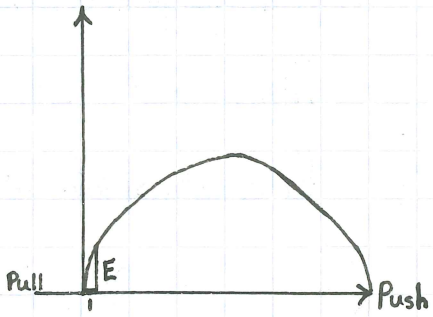
Glass



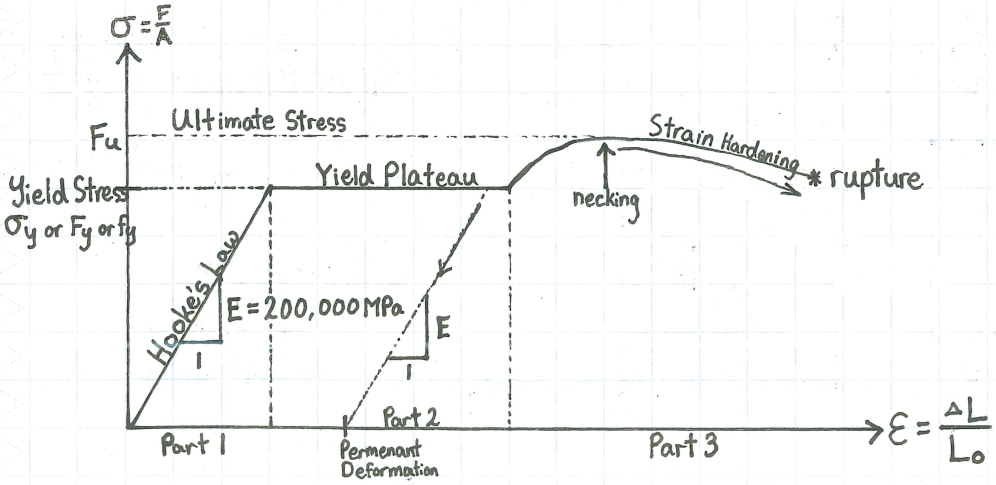
Brass

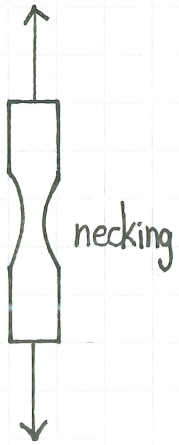


Concrete



2) Stress - Strain of Mild Steel



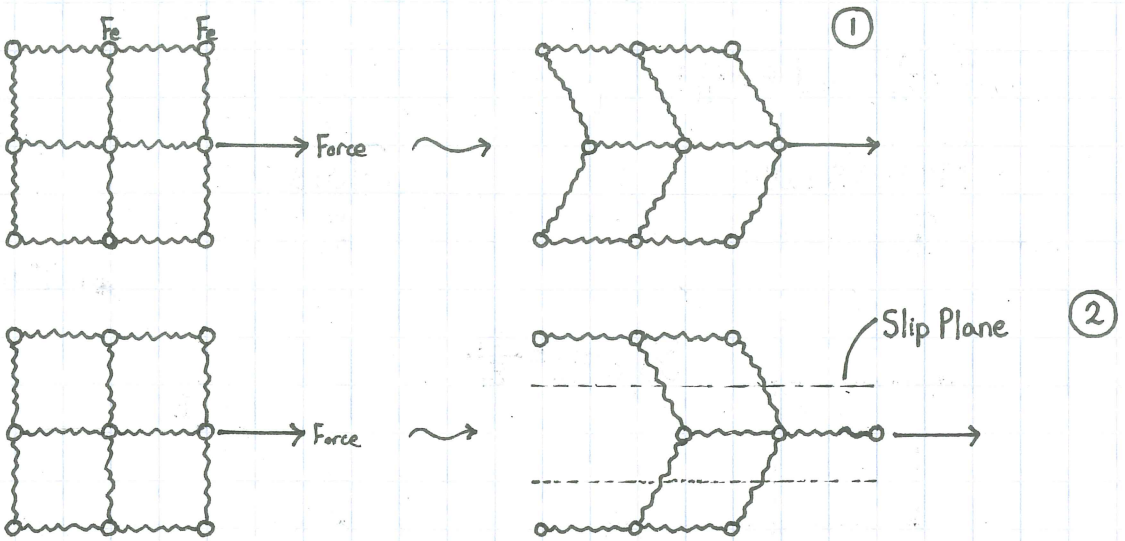


① Hooke's Law applies ⟶ linearly elastic
- no permanent damage or deformation
- $\sigma \leq F_y$
- design for this region

② Yield Plateau
- Material is being damaged ⟶ Permanent
- $\sigma = F_y$

③ Strain Hardening
- $F_y < \sigma < F_u$

3) Why?



4) Strain Energy

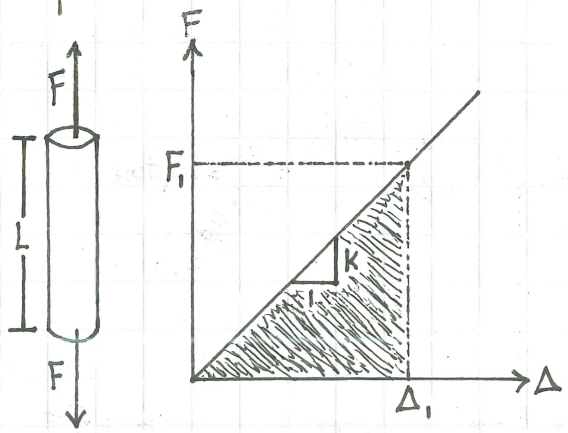
Strain energy is a form of potential energy you get if you deform a solid.

High School: work = Force · distance

for a material test: work = $\sum_i \text{Force}_i \cdot \Delta_i$ i = small steps

or Work = Energy = $\int F \, d\Delta$ = area under a load-displacement curve

Example



Strain Energy = $\frac{\Delta_1 F_1}{2}$ $(F_1 = K \cdot \Delta_1)$

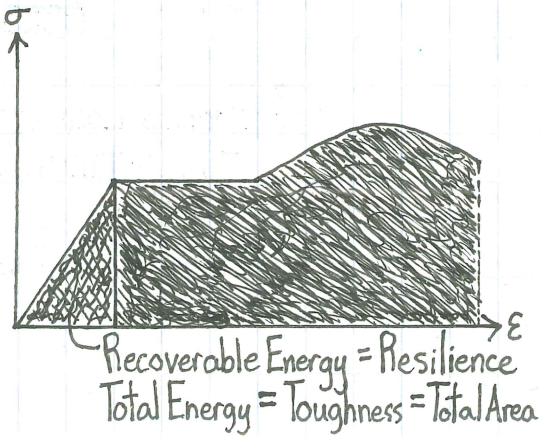
Energy = $\frac{\Delta_1 \cdot K \cdot \Delta_1}{2} = \frac{K(\Delta_1)^2}{2}$

Units = kN · mm
$= 10^3 \text{N} \cdot 10^3 \text{m}$
= Nm = Joule

Normalize by Volume
= Energy Density = Specific Energy

$\frac{J}{m^3} \rightsquigarrow \frac{MJ}{m^3}$

= Area Under σ-ε Curve



CIV102 - STRUCTURES and MATERIALS

Topic: Safety!

1) Types of Loads

a) Dead Loads (Things that can't be moved)

- Have low uncertainty
- Examples: Selfweight of buildings and bridges, fixed seatings, lamp posts, stuff behind the ceiling

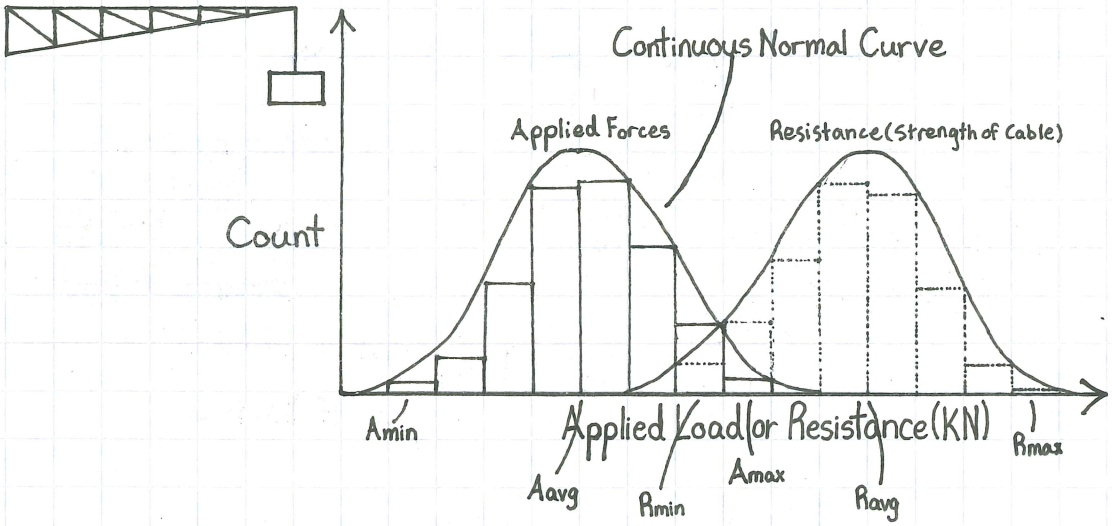
b) Live Loads (Things that can be moved)

- Have high uncertainty
- Examples: People, cars, water in pools, wind + snow

c) Others

- Earthquake load
- Thermal effects

Example



Concept of Safety

Limit States Equation:

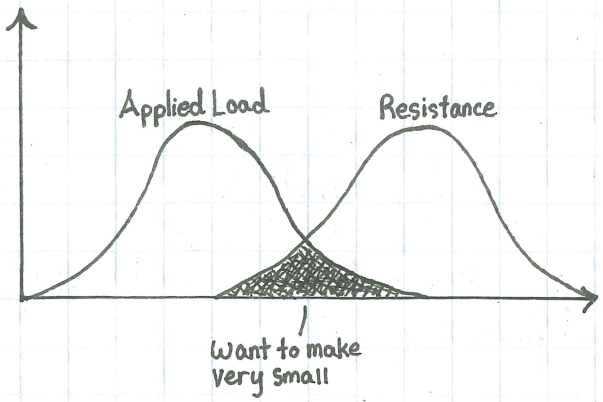
$$\text{Applied Load} \leq \text{Resistance}$$

$$A \leq R$$

or $$A_{max} \leq R_{min}$$

Reliability analysis to decide if things are safe.

Redraw Graph

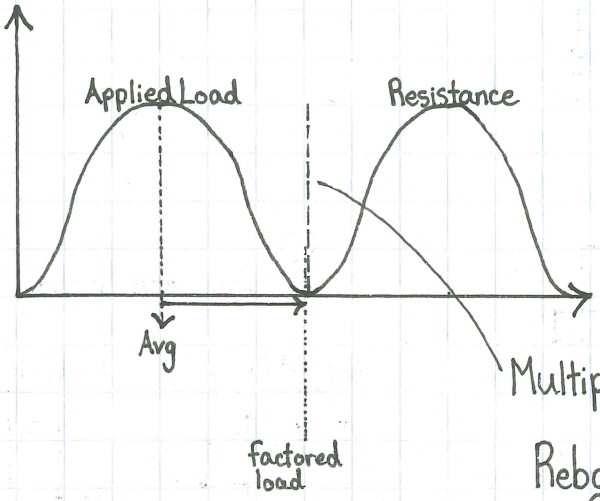


Area = Danger Zone where A > R = Collaps

∴ minimize area

2) How to account for this?

i) Partial Safety Factor Method



Multiply by $\phi < 1.0$
Steel $\phi = 0.9$
Rebar Steel $\phi = 0.85$
Concrete $\phi = 0.65$

Multiply Average Load By $\begin{cases} 1.25 \text{ if dead load} \\ 1.50 \text{ if live load} \end{cases}$

ii) Allowable Stress Design

Safety Factor = $\frac{R_{avg}}{A_{avg}}$

Brooklyn Bridge (1888) ⟶ SF = 5.0
Golden Gate Bridge (1937) ⟶ SF = 2.68
Akashi-Kaikyo Bridge (1998) ⟶ SF = 2.25

3) How to use allowable stress design

$\sigma_{allowable} = \frac{\sigma_{strength}}{FOS}$ FOS = Factor of Safety

$= \frac{f_y}{FOS} = \frac{\sigma_{yield}}{FOS}$

What FOS to use?

$FOS = \begin{cases} 2.0 \rightarrow \text{Use if we have warning of failure} \\ 3.0 \rightarrow \text{Use if we have no warning of failure} \end{cases}$

CIV102 - STRUCTURES and MATERIALS

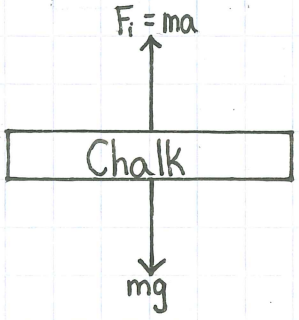
Topic: Oscillating Bodies

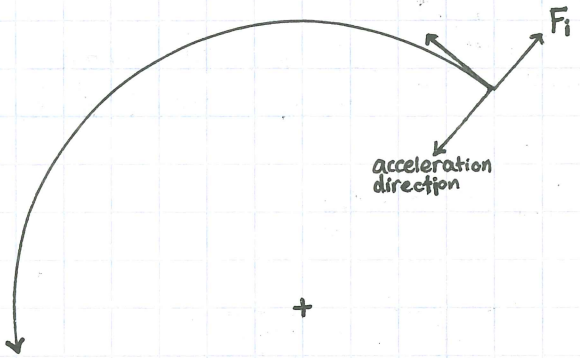
1) Oscillating Systems

Mechanical Oscillating Systems
Electrical Oscillating Systems
Electromagnetic Oscillating Systems

2) Dynamic Equilibrium

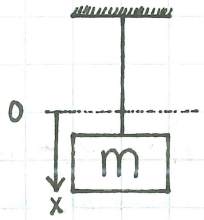
D'Alembert's Principle: "An accelerating system where the sum of the forces does not equal zero, can be placed in dynamic equilibrium by introducing an Inertial Force = $F_i = ma$ acting in a direction opposite to acceleration direction.





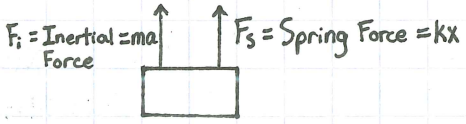
3) Free Vibration

- Ignore Gravity



Assume is accelerating downwards.

FBD



$$\sum F_x = 0$$
$$0 = -F_s - F_i$$
$$= F_s + F_i$$
$$0 = kx + mg$$

$$\text{Velocity} = \frac{dx}{dt} = v$$

$$\text{Acceleration} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = a$$

$0 = kx + m\frac{d^2x}{dt^2}$ differential equation

Guess + Check

$X(t) = A\sin(\omega_n t + \Phi)$ (A: Scale Factor, Φ: Phase Shift, ω_n: Natural Frequency)

$\frac{dx}{dt} = A\omega_n \cos(\omega_n t + \Phi)$

$\frac{d^2x}{dt^2} = -A\omega_n^2 \sin(\omega_n t + \Phi)$

Substitute into differential equation

$0 = kx + m\frac{d^2x}{dt^2}$

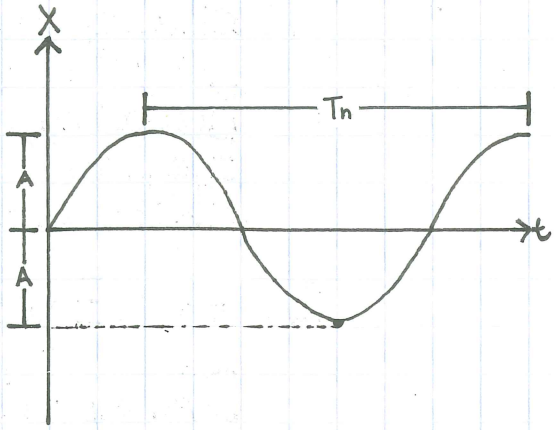
$0 = k \cdot A\sin(\omega_n t + \Phi) - m \cdot A\omega_n^2 \sin(\omega_n t + \Phi)$

$0 = k - m\omega_n^2$

$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$ ω_n = natural frequency $\left[\frac{rad}{sec}\right]$ ← rate it wants to vibrate at

Convert $\omega_n \left[\frac{rad}{sec}\right]$ into $f_n = Hz = \left[\frac{Cycles}{Second}\right]$

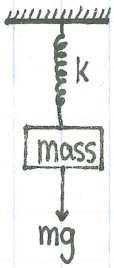
$f_n = \frac{1}{2\pi} \cdot \omega_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$



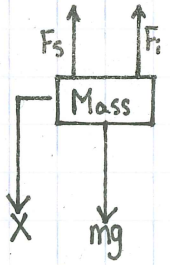
T_n = Natural Period $= \frac{1}{f_n} = 2\pi\sqrt{\frac{m}{k}}$

$\boxed{T_n = 2\pi\sqrt{\frac{m}{k}}}$

4) Add in gravity



FBD



$\Sigma F_x = 0$

$0 = mg - F_i - F_s$

$0 = m\frac{d^2x}{dt^2} + kx - mg$

Guess + Check

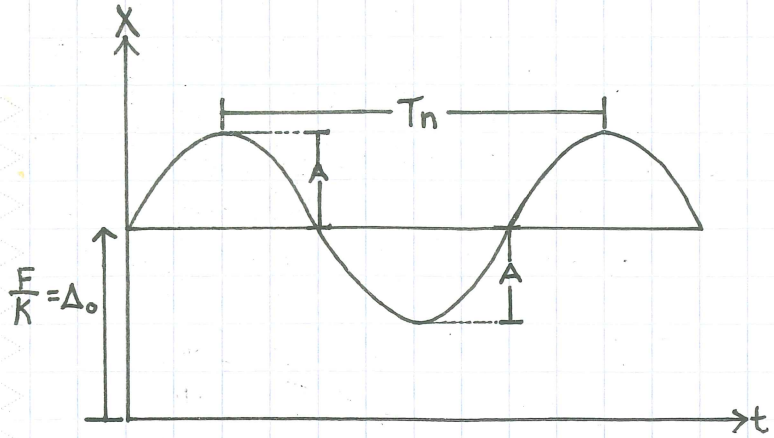
guess $X(t) = A\sin(\omega_n t + \Phi) + \Delta_o$

Δ_o = Static displacement caused by load mg

$F = k\Delta_o$

$\boxed{\Delta_o = \frac{F}{k}}$

Plot It



Simpler form for gravity-involved forces

$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ $\quad \Delta_o = \frac{mg}{K}$, $K = \frac{mg}{\Delta_o}$

$f_n = \frac{1}{2\pi}\sqrt{\frac{mg}{\Delta_o m}}$ if Δ_o in mm, $g = 9810 \frac{mm}{s^2}$

$f_n = \frac{1}{2\pi}\sqrt{\frac{9810}{\Delta_o}}$

$$\boxed{f_n = \frac{15.76}{\sqrt{\Delta_o}}}$$

Demonstration $\quad \Delta_o = 320mm$

$f_n = \frac{15.76}{\sqrt{320mm}} = 0.88 Hz$ (Theoretical)

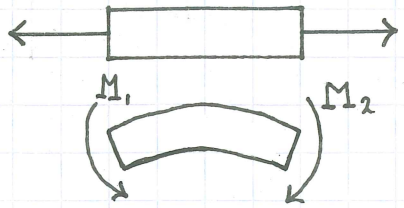
$\frac{10 \text{ cycles}}{11.7} = 1.171 \frac{sec}{cycles}$

$= 0.854 Hz$ (Experimental)

CIV102 - STRUCTURES and MATERIALS

Topic: What is I?

1) So far



Statics

$\Sigma F_x = 0$

$\Sigma M = 0$

Dynamics

$F = ma$ — linear acceleration $\left[\frac{m}{s^2}\right]$

mass

$M = I_m \cdot \alpha$ — rotational acceleration $\left[\frac{rad}{sec^2}\right]$

I_m: Mass moment of Inertia

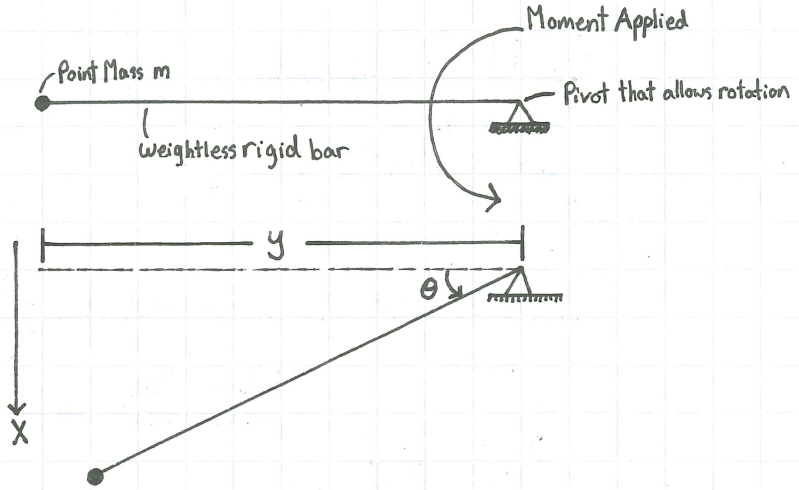
Mass moment of Inertia = I_m

Second Moment of Area = I

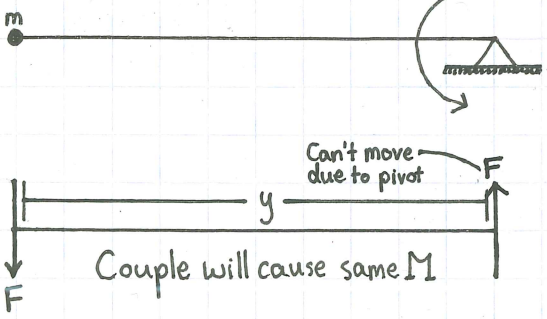
2) I_m = ?

Assumptions:

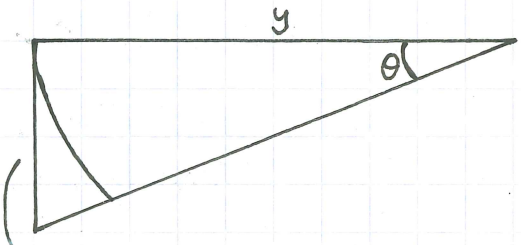
- Mass is a point mass
- Angles are small
- One mass



Forces



Geometry



$\cong \theta \cdot y$ = arclength which almost equals linear displacement

$M = F \cdot y$

$F = \frac{M}{y}$

	Variable	Linear Distance	Linear Velocity	Linear Acceleration
Linear	X	X	$\frac{dx}{dt}$	$\frac{d^2x}{dt^2}$
	θ	θy	$y\frac{d\theta}{dt}$	$y\frac{d^2\theta}{dt^2}$

Apply $F = ma$

$$\frac{M}{y} = my\frac{d^2\theta}{dt^2}$$

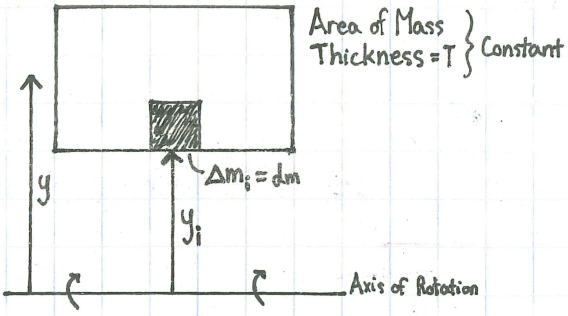
$$M = m \cdot y^2 \cdot \frac{d^2\theta}{dt^2} = \alpha$$

$$M = (my^2)\alpha$$

3) Add more masses

$$\boxed{I_m = \sum_{i=1}^{n} m_i \cdot y_i^2}$$

4) Masses with volume (not point masses)



$$I_m = \sum m_i \cdot y_i^2$$

take limit as $\Delta m_i \longrightarrow 0$

$$\boxed{I_m = \int y^2\, dm}$$

5) Second Moment of Area = I

· Property of a cross section

Lets define ρ = mass density of material

Mass of Object $= \rho \cdot \text{Area} \cdot t$

$$I_m = \int y^2\, dm \quad (dA = \text{Area})$$

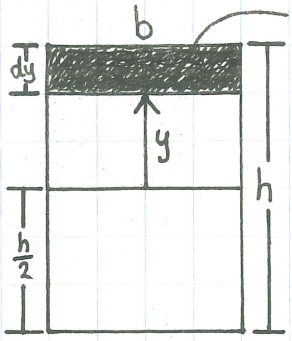
$$= \int y^2 \cdot \rho \cdot t\, dA$$

$$I_m = \rho \cdot t \boxed{\int y^2\, dA}$$

define this as second moment of area; I [mm^4] or moment of inertia

$$I_m = \rho \cdot t \cdot I$$

6) I for a rectangle rotating about its centroid



$$I = \int y^2 \, dA$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 \, b \cdot dy$$

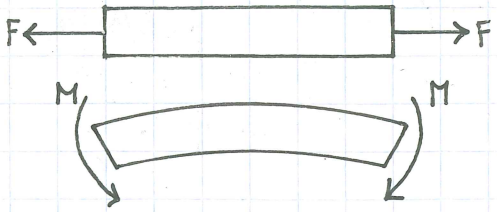
$$I = \frac{by^3}{3}\Big|_{-\frac{h}{2}}^{\frac{h}{2}} = \left[\frac{b \cdot h^3}{2^3 \cdot 3} - \left(b\left(\frac{-h}{2}\right)^3 \cdot \frac{1}{3}\right)\right]$$

$$\boxed{I = \frac{bh^3}{12}}$$

CIVI02 - STRUCTURES and MATERIALS

Topic: Bending of Beams

1) Bending



$$\begin{cases} F = (\Delta L)K \\ \sigma = E\varepsilon \end{cases}$$

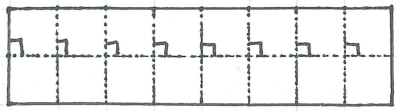
$$M = \frac{1}{\text{Relation betwixt moment and bentness}} \cdot \frac{1}{\text{Measure of "bentness"}}$$

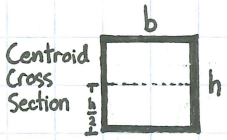
Assume

Hooke's Law

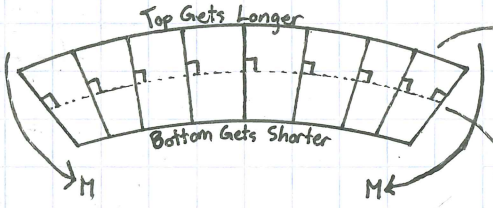
"Plane Sections Remain Plane" → Euler (Swiss), Bernoulli, Navier (French), Timochenko, Hooke's (English, 1678)

Undeformed Shape





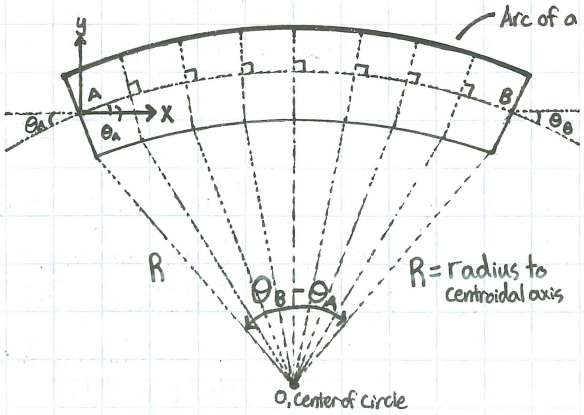
Deformed Shape



arc of a circle

length at centroidal axis does not change

2) Flexural Stiffness



Note: Diagram Not to Scale. Lines should be straight.

L_{AB} = Length Along Arc

$L_{AB} = (\theta_B - \theta_A) \cdot R$

From Hooke, we will assume this is constant.

If θ_A and θ_B are small, then the horizontal component of length L_{AB} will equal L_{AB}.

Define degree of "bentness" as curvature

$$\phi = \frac{(\theta_B - \theta_A)}{L_{AB}}$$

= Rate of Change of Angle = Slope of Beam

$$\phi = \frac{d\theta}{dx}$$

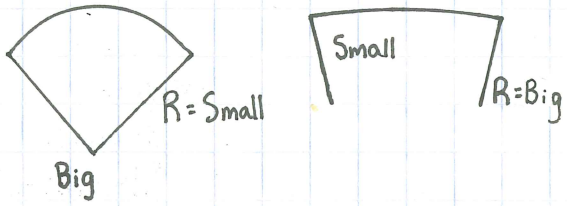
$$\phi = \underbrace{\frac{(\theta_B - \theta_A)}{L_{AB}}}_{\text{Whole Thing}} = \underbrace{\frac{(\theta_B - \theta_A)}{l_{AB}}}_{\text{Small Parts}}$$

$$L_{AB} = (\theta_B - \theta_A) \cdot R$$

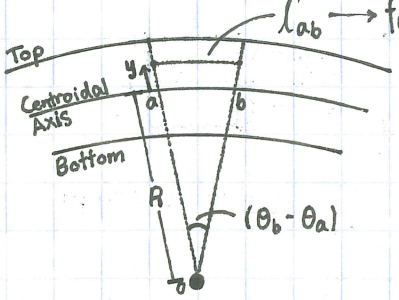
$$R = \frac{L_{AB}}{(\theta_B - \theta_A)}$$

$$\phi = \frac{(\theta_B - \theta_A)}{L_{AB}}$$

$$\phi = \frac{1}{R}$$



3) Flexural Strains



$l'_{ab} \neq l_{ab}$

$$l'_{ab} = l_{ab}$$

$$l'_{ab} = (\theta_b - \theta_a) \cdot (R + y)$$

$$= (\theta_b - \theta_a) \cdot \left(\frac{1}{\phi} + y\right)$$

$$\phi = \frac{(\theta_b - \theta_a)}{l_{ab}} \longrightarrow (\theta_b - \theta_a) = \phi l_{ab}$$

$$l'_{ab} = \phi l_{ab}\left(\frac{1}{\phi} + y\right)$$

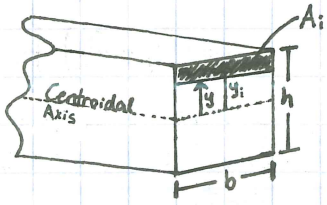
$$l'_{ab} = l_{ab} + \phi l_{ab} y$$

$$\varepsilon = \frac{\Delta L}{L}, \quad \Delta L = l'_{ab} - l_{ab}$$

$$\varepsilon = \frac{\phi l_{ab} \cdot y}{l_{ab}}$$

$$\boxed{\varepsilon = \phi \cdot y}$$

4) Relation of M to ϕ



Strain $= \varepsilon_i = \phi y_i$

Stress $= \sigma_i = E \phi y$

Force $= F_i = E \phi y_i A_i$

Moment About Centroid = Force · Lever Arm

Moment Component $= E \phi y_i A_i y_i$

Total Moment $= \sum_{i=1}^{n} E \phi y_i^2 \cdot A_i$

take limit as $A_i \rightarrow 0$

$$M = \int E \phi \cdot y^2 \, dA$$

$$= E \phi \underbrace{\int y^2 \, dA}_{\text{Second Moment of Area} = I}$$

$$M = E \phi I$$

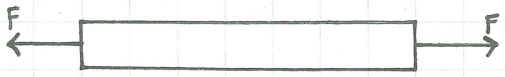
$$\boxed{M = EI\phi}$$

CIV102 - STRUCTURES and MATERIALS

Topic: Analysis of Structures

1) Structural Members = Components

So far



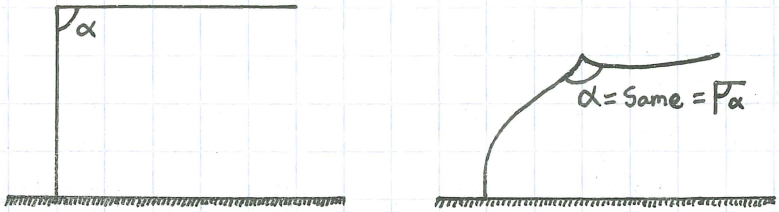
Now

N = Axial Load, kN
↳ Aligned with Axis
V = Shear Force, kN
↳ ⊥ to longitudinal axis
M = Internal Bending Moment, kN·m

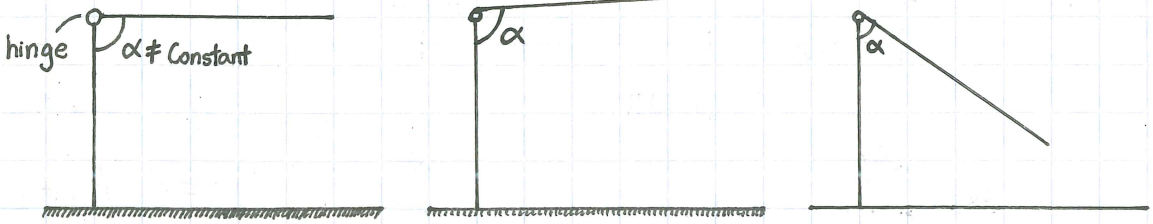
} Internal Forces

2) Connect Components

i) Rigid Connection (Connection = Where they touch)

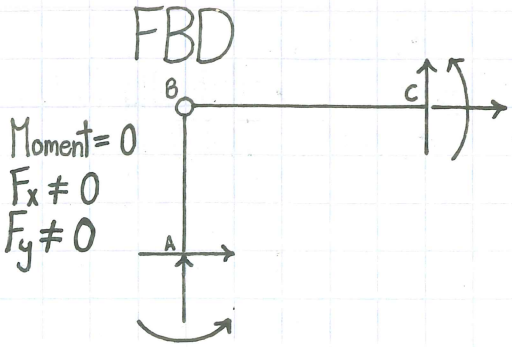


ii) Hinged Connection (Internal Bending Moment at the Hinge = Zero kN·m by Definition)



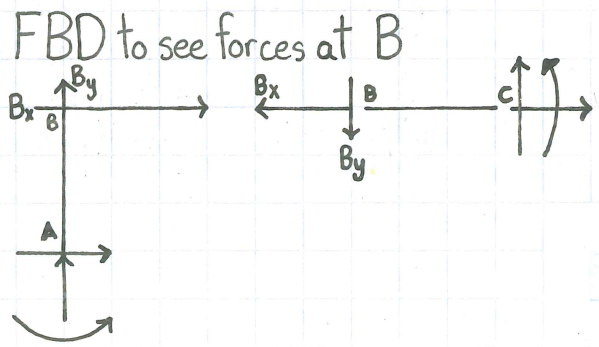
Forces at Hinge

FBD



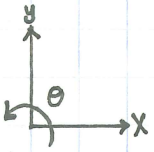
Moment = 0
$F_x \neq 0$
$F_y \neq 0$

FBD to see forces at B

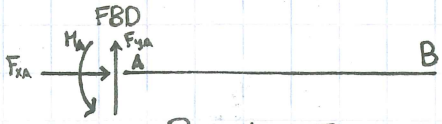


3) Connecting to the Ground (Boundary Conditions/Supports)

i) Fixed End — Fully Connected to Ground



(Cantilevers)

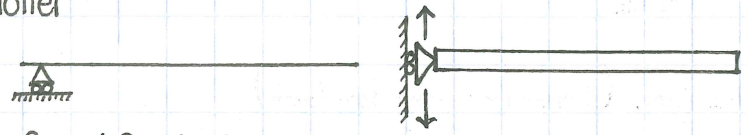


Reaction Forces = Ground Pushing on the Structure

ii) Pin

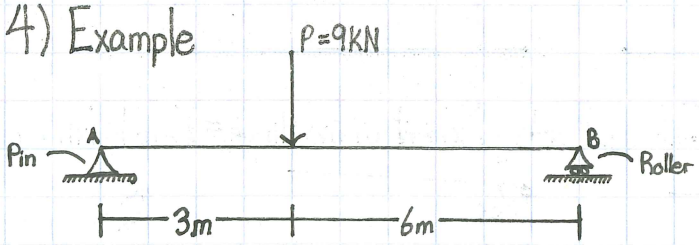
Resist Force in X = F_x
Resist Force in y = F_y
Cannot Resist a Moment

iii) Roller



Cannot Resist a Moment
Cannot Resist a Horizontal Force
Can Resist a F_y Force

4) Example

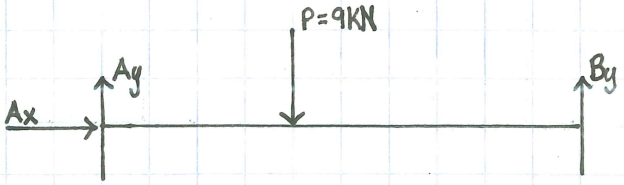


$\Sigma F_x = 0$

$0 = A_x$

$\Sigma M_A = 0$

$0 = -9kN \cdot 3m + B_y \cdot 9m$

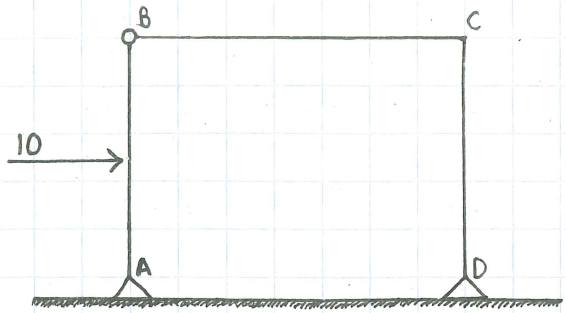


$\Sigma F_y = 0,\ A_y = 6kN$

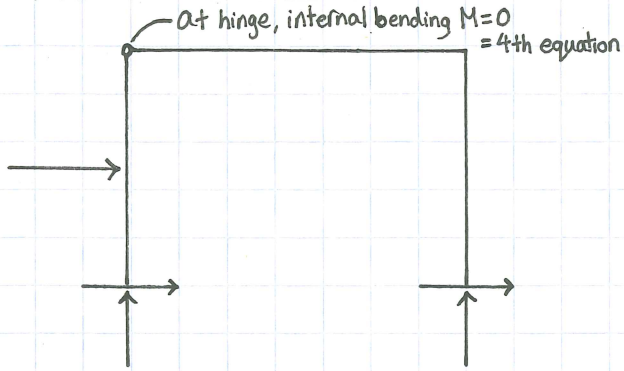
5) Determinacy

i) Fewer Unknowns than Equations ⟶ Unstable
ii) Same Number of Equations as Unknowns ⟶ Statically Determinate
iii) More Unknowns than Equations ⟶ Statically Indeterminate

6) Example with Internal Hinge



FBD

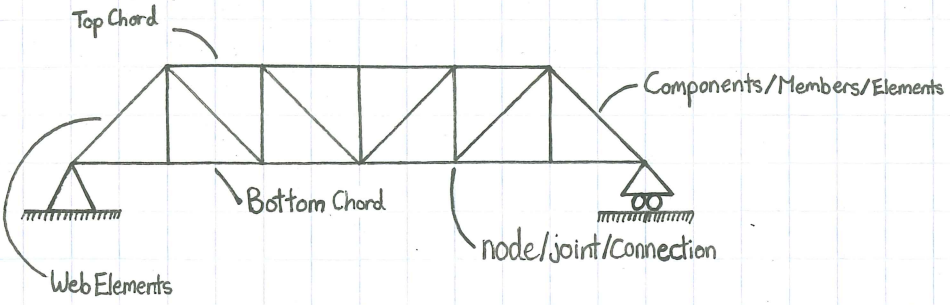


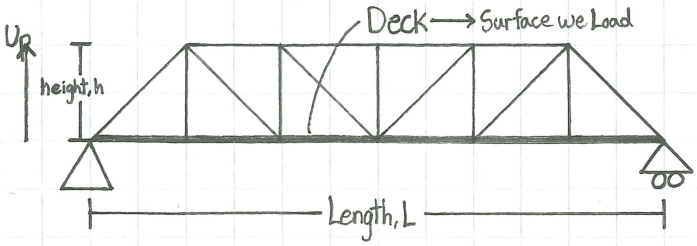
CIV102 - STRUCTURES and MATERIALS

Topic: Trusses

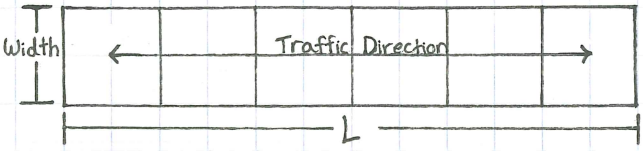
1) Truss

- Stiff, Strong, Light





Elevation View (From Side)



Plan View (From Above)

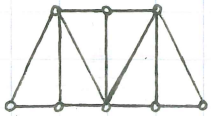
2) Process of Design of a Bridge

i) Select Geometry
ii) Determine Applied Loads
iii) Analyze the Forces Inside the Members
iv) Select Member Sizes
v) Calculate Displacements
vi) Check Dynamic Characteristics
vii) Iterate
viii) Detailed Design

} Preliminary Design

3) Analysis Assumptions for Trusses

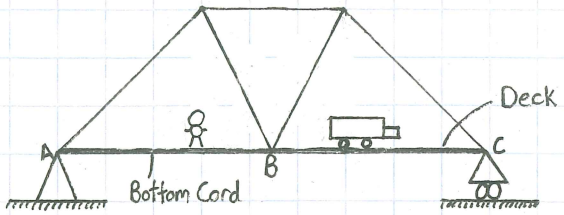
i) Small Displacements
ii) All Connections are Assumed to be Hinges
- Concequence: All members are subjected to tension or compression only (Shear = 0, Moment = 0)



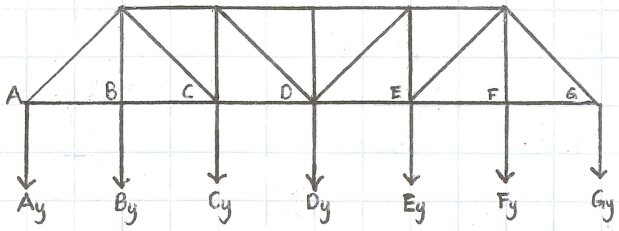
∴ Internal Bending M = Zero at all joints

iii) All Loads are Applied at Nodes

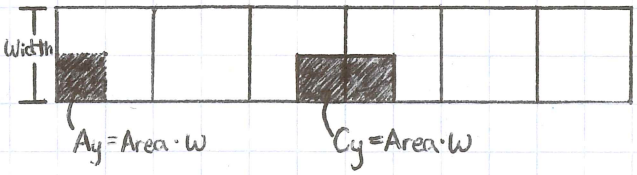
Deck Details



Loads on the Nodes ⟶ How to Calculate
- Tributary Area
- Uniform Load w $\left[\frac{kN}{m^2}\right]$



Elevation



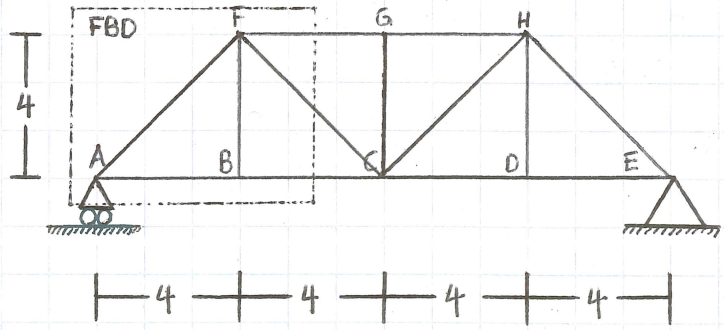
Plan

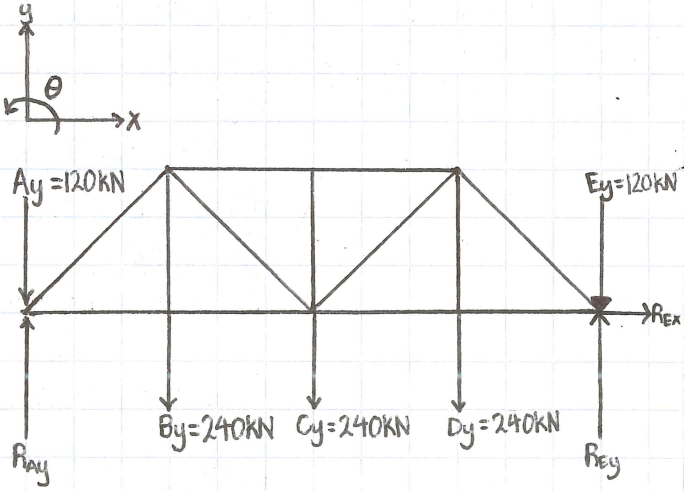
4) Method of Sections

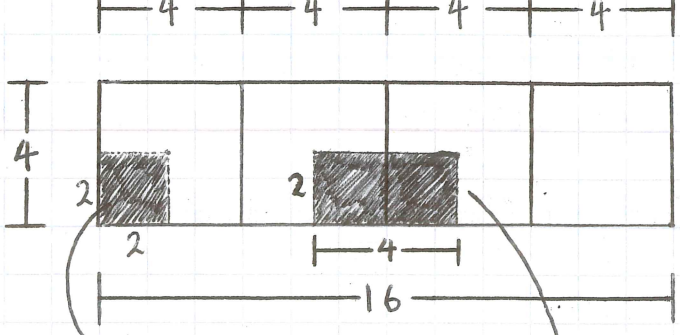
- Draw a FBD of Part of Structure
- I can only allow 3 unknown forces since I have 3 equations:

$\Sigma F_x = 0$
$\Sigma F_y = 0$
$\Sigma M = 0$

5) Example $w = 30\frac{kN}{m^2}$







$\Sigma F_x = 0 \quad R_{Ex} = 0$

$\Sigma M_A = 0$

$0 = -240 \cdot 4 - 240 \cdot 8 - 240 \cdot 12 - 120 \cdot 16 + R_{Ey} \cdot 16$

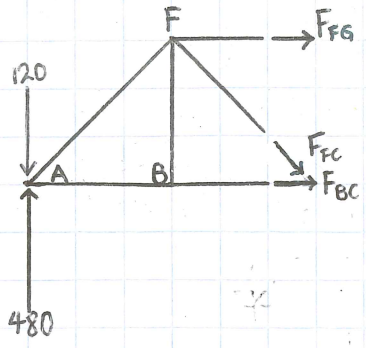
$R_{Ey} = 480kN$

$\text{Area} = 2 \cdot 2 = 4m^2$

$Ay = 4m^2 \cdot 30\frac{kN}{m^2} = 120kN$

$Cy = 4 \cdot 2 \cdot w = 240kN$

What is F_{FG}=? F_{FC}=? F_{BC}=?



$\sum M_F = 0$

$0 = 120 \cdot 4 - 480 \cdot 4 + F_{BC} \cdot 4$

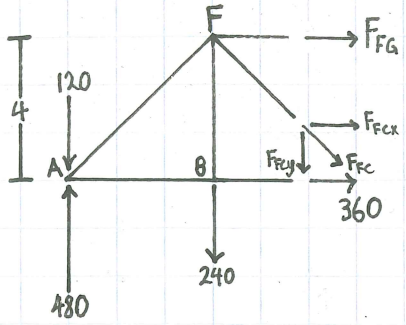
$F_{BC} = 360\,KN$ tension

$\left.\begin{matrix}\sum F_x = 0 \\ \sum F_y = 0\end{matrix}\right\}$ To get others

CIV102 - STRUCTURES and MATERIALS

Topic: Method of Joints

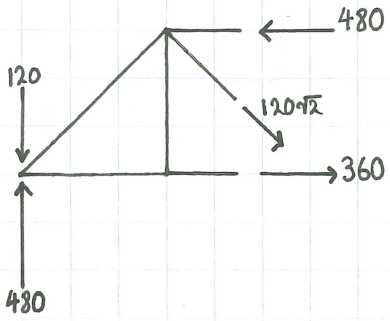
1) From Last Time



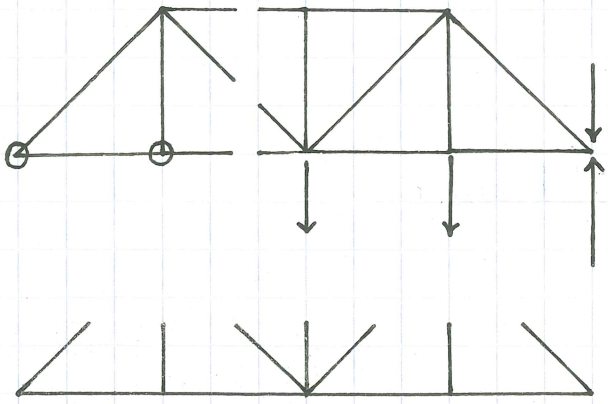
$\Sigma F_x = 0$
$\Sigma F_y = 0$
$\Sigma M = 0$

$\Sigma F_y = 0$
$0 = 480 - 120 - 240 - F_{FCy}$
$F_{FCy} = 120KN$ (tension)

$F_{FCx} = F_{FCy} = 120KN$ Since F_{FC} @ 45°



$\Sigma F_x = 0$
$0 = F_{FG} + 120 + 360$
$F_{FG} = -480KN$ (Compression)

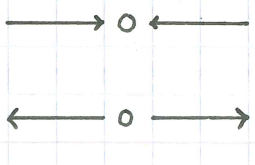


7 Unknowns

2) Method of Joints

Do an FBD @ each joint

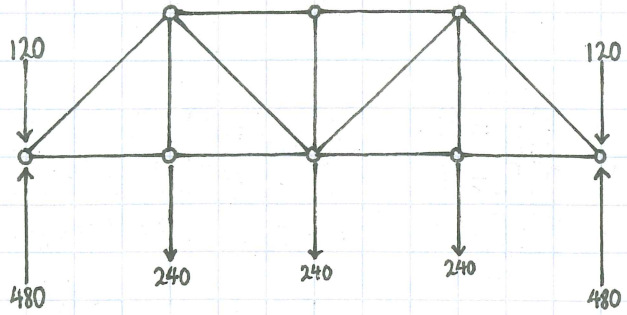
- The pin in the hinge is what the freebody diagram is of



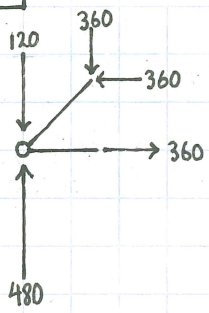
Since all forces go through the center of the pin $\Sigma M = 0$ tells us only $0 = 0$

Only 2 equations left $\Sigma F_x = 0$
$\Sigma F_y = 0$

Example: Same Truss

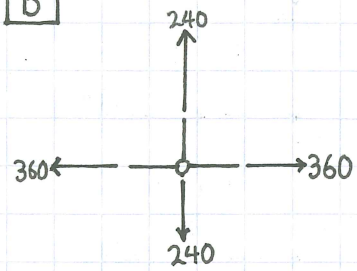


A

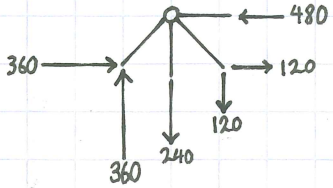


$F_{Diag} = 360\sqrt{2}$

B



F



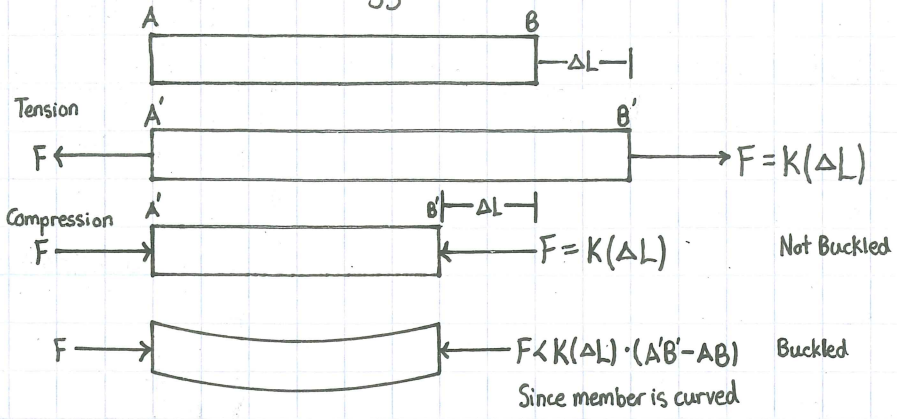
Exam (Asymmetrically Loaded Truss)

CIV102 – STRUCTURES and MATERIALS

Topic: Buckling

1) Why do things buckle?

- Achieve a lower energy state than if it had not buckled



Euler 1757

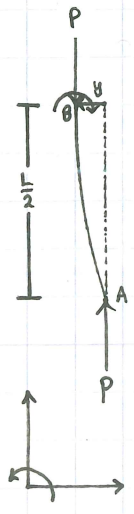
$$P_E = P_{cr} = P_{critical}$$

(P_E: $_{Euler}$)

2) Derivation

Assumptions

- EI = Constant Along Length
  ($M = EI \cdot \Phi$)
- Ability to rotate @ top and bottom
  Column (Pin or Roller)
- Perfectly straight
- Small Displacements



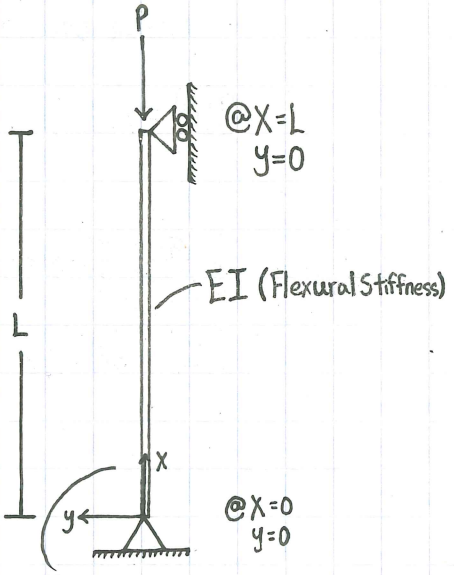
$$\Sigma M_B = 0$$
$$0 = -M + P \cdot y$$
$$M = P \cdot y \text{ @ Midheight}$$

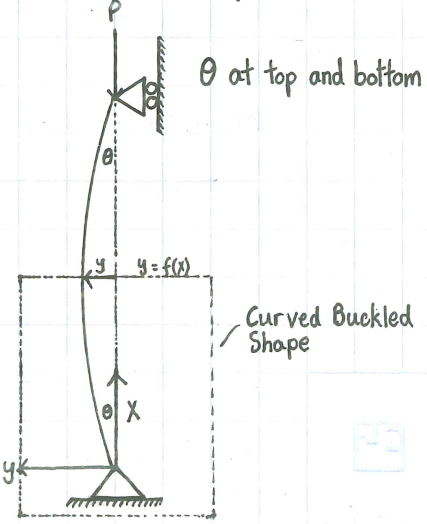
In general:
$$M(X) = P \cdot y(X)$$

$$\Phi = \text{Curvature} = -\frac{d\theta}{dx} = \frac{-d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d^2y}{dx^2}$$

$$M = P \cdot y(X), \quad M = EI\Phi$$
$$P \cdot y(X) = EI\Phi$$

Differential Equation:
$$\boxed{P \cdot y = EI\frac{d^2y}{dx^2}}$$





Guess Answer

$y(x) = A\sin(\omega x + b)$

$P \cdot y = -EI\frac{d^2y}{dx}$

$y(x) = A\sin(\omega x + b)$

$\frac{dy}{dx} = A\omega\cos(\omega x + b)$

$\frac{d^2y}{dx^2} = -A\omega^2\sin(\omega x + b)$

Substitute in

$P \cdot A\sin(\omega x + b) = -EI(-A\omega^2\sin(\omega x + b))$

$P = EI\omega^2$

$\omega = \sqrt{\frac{P}{EI}}$

Apply boundary conditions @ $X=0, y=0$

$y(0) = A\sin(\omega x + b) = 0 \rightarrow A = 0$ (trivial case)

$b = 0$

@ $X=L, y=0$

$y(L) = A\sin(\omega L) = 0$

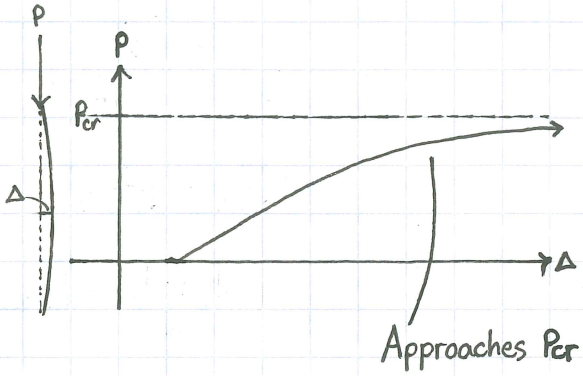
$\therefore \omega L = \pi, 2\pi, 3\pi, 4\pi, \cdots, n\pi$

$\omega = \sqrt{\frac{P}{EI}}, \quad \omega = \frac{n\pi}{L}$

$\frac{n\pi}{L} = \sqrt{\frac{P}{EI}}$

$$P_{cr} = \frac{\pi^2 EI n^2}{L^2} \qquad n = 1, 2, 3, 4, 5 \ldots$$

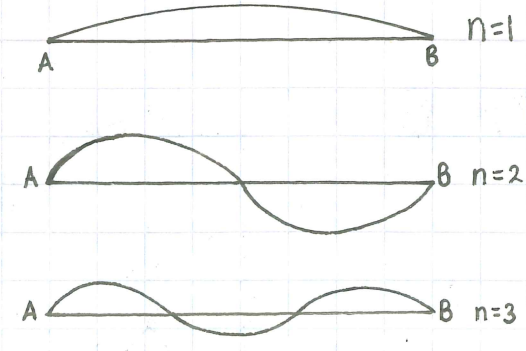
Euler buckling load



Safest Case = Practical Case $(n=1)$

$P_{Critical} = \frac{\pi^2 EI}{L^2}$

$A\sin(\omega x + b)$



CIV102 - STRUCTURES and MATERIALS

Topic: Design of Truss Members

1) $P_{cr} = \frac{\pi^2 E I}{L^2}$

i) Estimate Self Weight
ii) Calculate the Joint-Loads from Tributary Area
iii) Calculate Reaction Forces
iv) Solve for Member Internal Forces {Joints / Sections}
v) Size the Cross Sections of Each Member to Safely Carry the Load

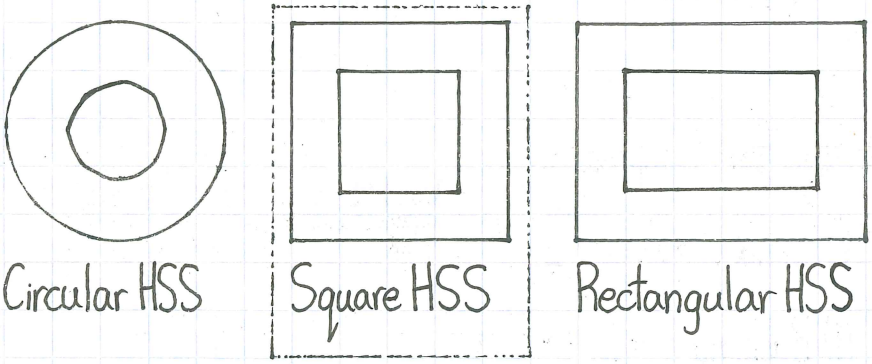
Size?

Tension $F_{fail} = A \cdot f_y$ (force, Strain)

Compression $F_{fail} = A \cdot f_y$ or $F_{fail} = \frac{\pi^2 E I}{L^2}$ (Size term) } Smaller will Govern

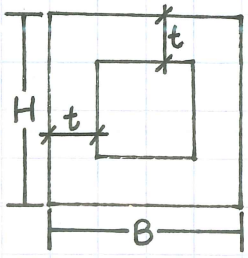
We Will Use HSS (Hollow Structural Section)

Steel, $E = 200{,}000\,MPa$ $f_y = 350\,MPa$



Circular HSS | Square HSS | Rectangular HSS

HSS Designation



HSS H x B x t

Example HSS 305 x 305 x 12.7

Why HSS?

- Light
- Strong (High Second Moment of Area)
- Stiff

I, A Combined

$r = \sqrt{\frac{I}{A}}$ radius of gyration

- Geometric Property
- Theoretical Idea

2) Design for Tension

From Method of Joints $\longrightarrow F_{demand}$

$$F_{capacity} \geq F_{demand}$$

$$A \cdot f_y \geq F_{demand}$$

$$F_{safe} \geq F_{demand}$$

$$\frac{A \cdot f_y}{FOS} \geq F_{demand}$$

$$\frac{A \cdot f_y}{2} \geq F_{demand}$$

$$A \geq \frac{2F_{demand}}{f_y}$$

3) Design for Compression

i) Yield $\rightsquigarrow A \geq \frac{2|F_{demand}|}{f_y}$ (Use FOS of 2 against compression yield)

ii) Buckling $\rightsquigarrow P_{cr} = \frac{\pi^2 E I}{L^2}$ $\quad P_{safe} = \frac{\pi^2 E I}{FOS\, L^2}$

For buckling FOS = 3.0 $\rightsquigarrow$ $\boxed{I \geq \frac{FOS\, L^2 F_{demand}}{\pi^2 E}}$

Stress at Buckling

$$\frac{P_{cr}}{A} = \frac{\pi^2 E I}{A \cdot L^2} \qquad r = \sqrt{\frac{I}{A}}$$

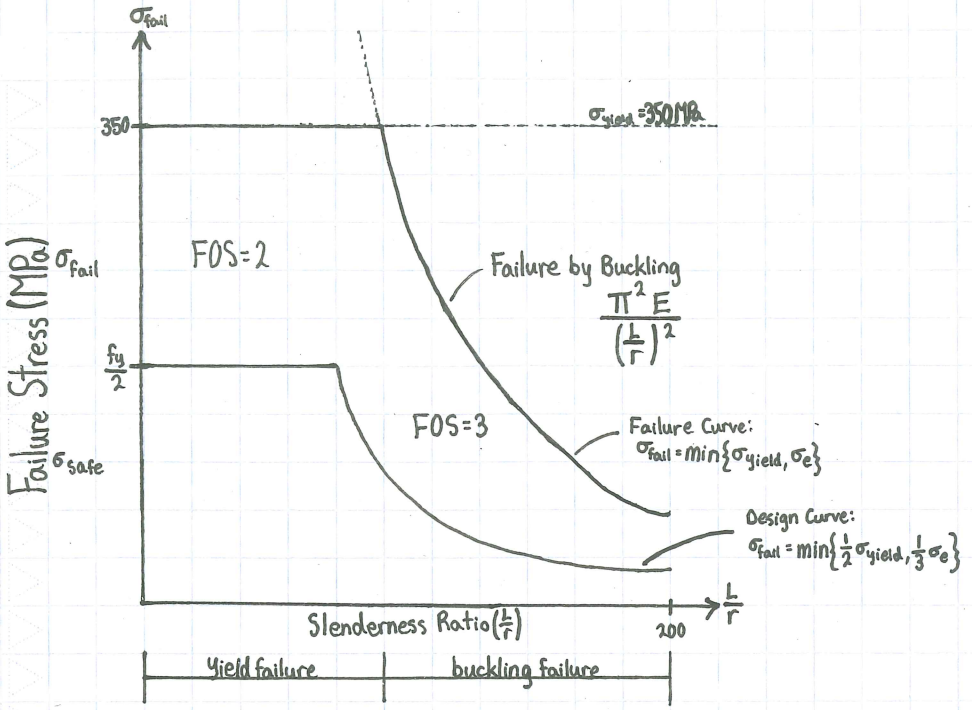
$$r^2 = \frac{I}{A}$$

$$\sigma_{Euler} = \frac{\pi^2 E r^2}{L^2}$$

$$= \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$$

$\frac{L}{r}$ = Slenderness Ratio

$\frac{L}{r} \leq 200$ always



Member Type	Cross-Sectional Area, A	Second Moment of Area, I	Radius of Gyration, r
Tension	$\geq \frac{2F_{demand}}{f_y}$	N/A	$\geq \frac{L}{200}$
Compression	$\geq \frac{2F_{demand}}{f_y}$	$\geq \frac{3.0F_{demand}L^2}{\pi^2 E}$	$\geq \frac{L}{200}$

Example $F_{demand}=250kN$ Compression

$L=5000mm$

$\sigma_y=f_y=350MPa$

$A \geq \frac{2 \cdot 250000}{350}$

$\geq 1429mm^2$

$I \geq \frac{3.0(250000)(5000^2)}{\pi^2(200000)}$

$\geq 9.5 \times 10^6 mm^4$

$r \geq 25mm$

Select HSS 152 x 152 x 4.8

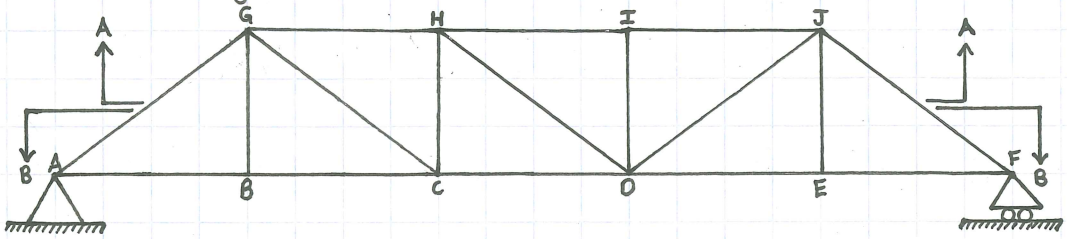
CIV102 - STRUCTURES and MATERIALS

Topic: Design for Wind

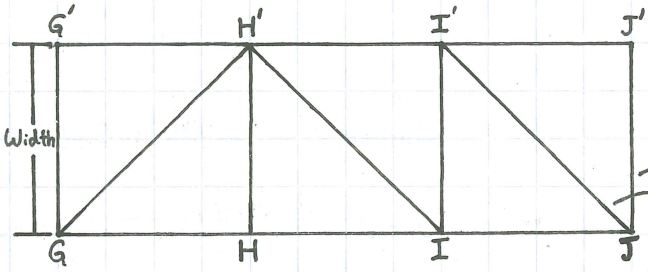
1) National Building Code of Canada (NBCC)

- Specifies load
- Floor loading for lecture hall
  - With fixed seats $= 2.4 \frac{kN}{m^2} = 50 \frac{lbs}{ft^2}$
  - Without fixed seats $= 4.8 \frac{kN}{m^2} = 100 \frac{lbs}{ft^2} = 100 psf$

2) Wind Design for Trusses

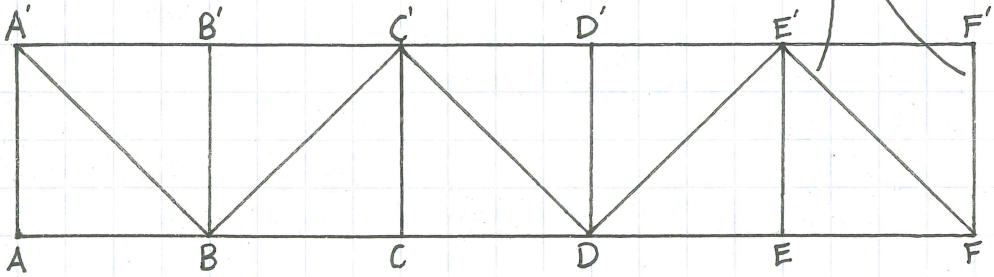


Elevation

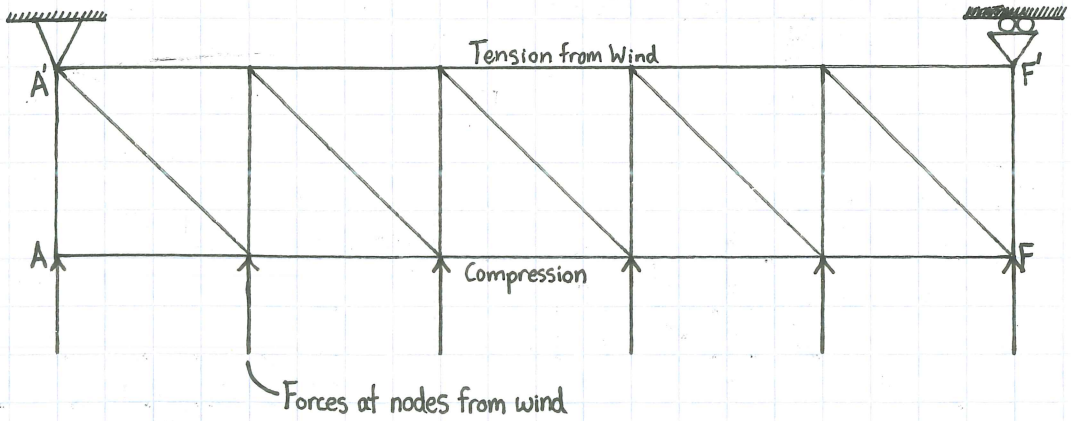


Plan A-A

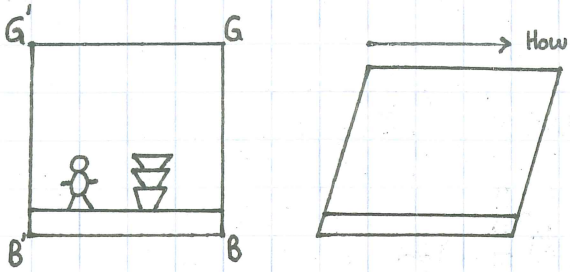
Wind Bracing

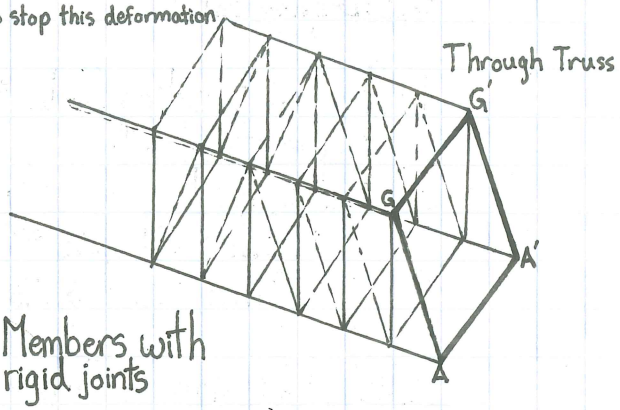


Plan B-B

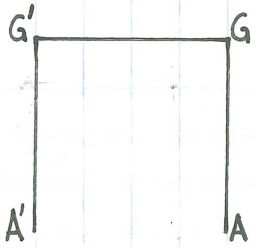


Usually only design diagonals and cross-bridge members for wind.
Neglect forces in bottom/top chord.

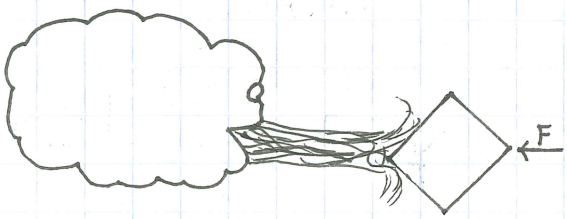




We will add a frame at each end = Members with rigid joints



3) Wind Forces



$W_{wind} \rightarrow$ Wind Pressure [kPa] or $[\frac{kN}{m^2}]$

$W_{wind} = \frac{1}{2} \rho \cdot V^2 \cdot C_D$

- ρ: Air Density
- V: Air Velocity
- C_D: Coefficient of Drag
  - Racing Car $C_D = 0.2$
  - Hindenburg $C_D = 0.08$
  - Sphere/Cylinder $C_D = 0.75$
  - Block $C_D = 1.5$

We will assume worst case senario

$\rho = 1.2 \frac{kg}{m^3}$

$C_D = 1.5$

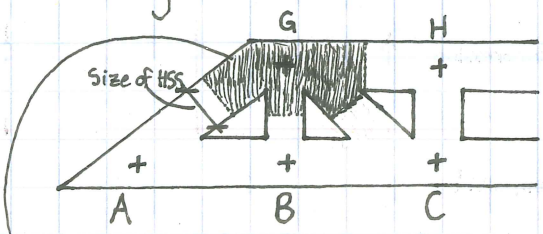
$V = \sim 100 mph, 170 \frac{km}{h} = 47.2 \frac{m}{s}$

$W_{wind} = \frac{1}{2}(1.2)(1.5)^2(47.2)$

$= 2 \frac{kN}{m^2}$

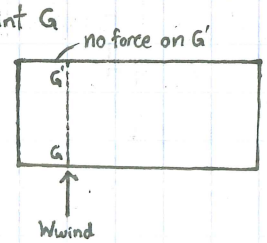
$= \underline{2 kPa}$

4) Tributary Area

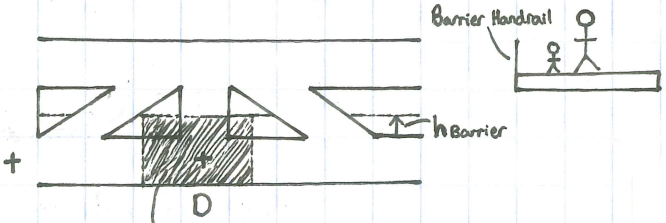


Tributary Area of Joint G

Force = Area · W_{wind}



Bottom Cord

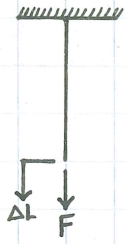


Tributary Area for D

CIV102 - STRUCTURES and MATERIALS

Topic: Δ

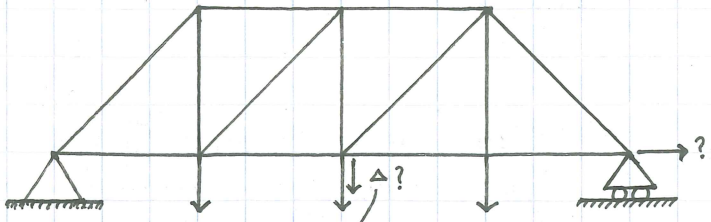
1) Displacements



$F = K \cdot \Delta L$

$F = \frac{AE}{L}(\Delta L)$

$$\Delta L = \frac{FL}{EA}$$



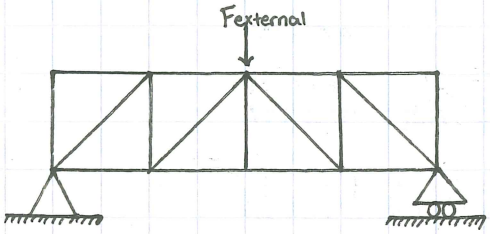
What is $\Delta_{allowable}$

$\Delta < \frac{L}{300} = \underline{OK}$

from live load only

We will use Energy!

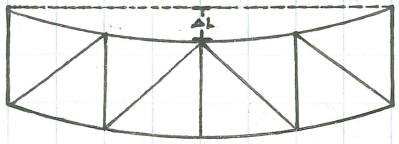
- Conservation of Energy



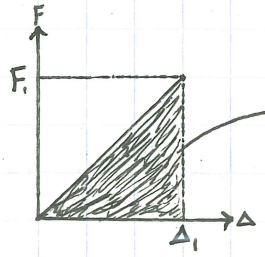
$W_{external}$ = Work done by external forces

$W_{internal}$ = Work absorbed by strain energy internally

$$\underline{W_{external} = W_{internal}}$$



Consider External Force/Work



Area Under Curve = Energy

$W_{ext} = \int F \, d\Delta$

$Area = \frac{F_1 \Delta_1}{2}$ ← Goal

$F_{external}$

Internal Forces + Work

For each element, i

$$\text{Strain Energy} = \frac{F_i \cdot \Delta_i}{2}$$

(F_i: Internal Force; Δ_i: change in length of element i)

$$\Delta_i = \frac{F_i L_i}{E_i A_i}$$

$$\text{Energy} = \frac{F_i \cdot F_i L_i}{E_i A_i 2}$$

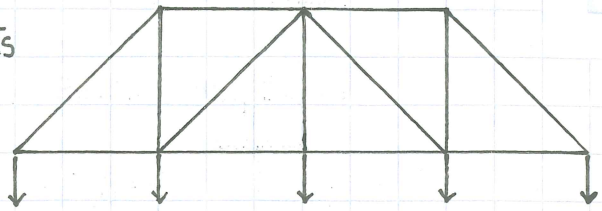
$W_{external} = W_{internal}$

$$\frac{1}{2} F_{external} \cdot \Delta_i = \sum_{i=1}^{n} \frac{F_i F_i L_i}{2 E_i A_i}$$

$F_{external}$: Known; Δ_i: goal; $\sum$: All known

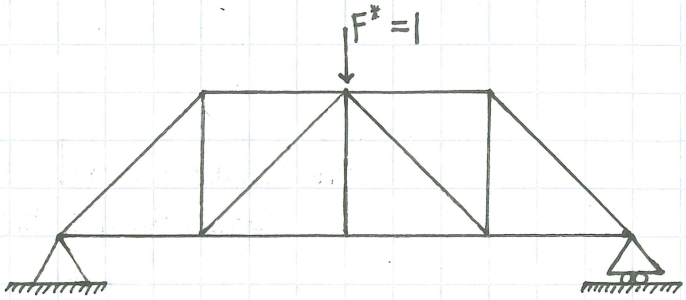
2) Virtual Work

- Real system will have actual displacements
- External Work/Force
- Internal Work/Force



Virtual System = Same Structure
- None of original loads included
- Apply a Force F^* at location of interest in direction of interest

F^* = Virtual Force = 1N = Unit Load



How to Apply Principle of Virtual Work (PVW)

1) For Real System = Real Applied Loads
- Solve for internal forces
- Solve for internal change in member lengths $\boxed{\Delta_i = \frac{F_i \cdot L_i}{E_i \cdot A_i}}$

2) Virtual System
- Remove real loads + Apply $F^* = 1N$
- Solve for internal Forces

Apply PVW

External VW = Internal VW

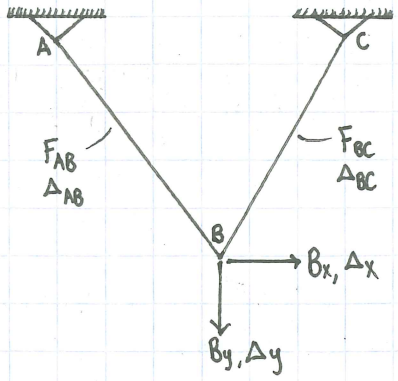
$F^* \cdot \Delta_{ext} = \sum F_i^* \cdot \Delta_i$ ($\frac{1}{2}$ factors are missing for virtual work)

Virtual — Real

$$F^* \cdot \Delta_{ext} = \sum F_i^* \cdot \frac{F_i L_i}{E_i A_i}$$

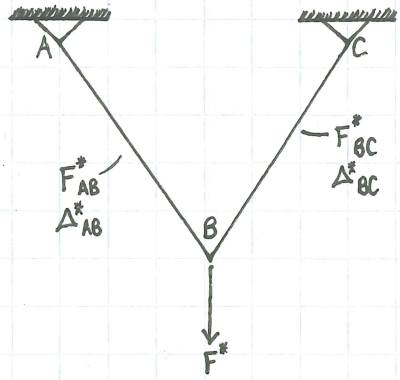
Why does it work?

Real



$$\frac{B_x \cdot \Delta_x}{2} + \frac{B_y \cdot \Delta_y}{2} = \frac{F_{AB} \cdot \Delta_{AB}}{2} + \frac{F_{BC} \cdot \Delta_{BC}}{2} \quad (1)$$

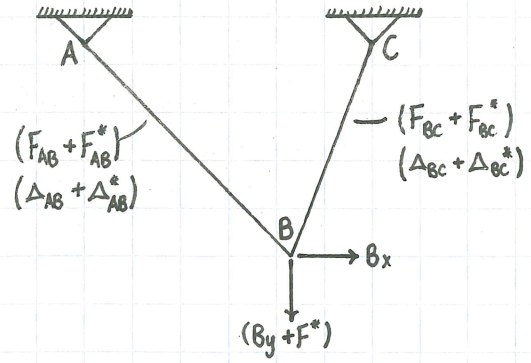
Virtual



$$\frac{F^* \cdot \Delta^*_y}{2} + \frac{B^*_x \cdot \Delta^*_x}{2} = \frac{F^*_{AB} \cdot \Delta^*_{AB}}{2} + \frac{F^*_{BC} \Delta^*_{BC}}{2} \quad (2)$$

$B^*_x = 0$

Combine Real + Virtual



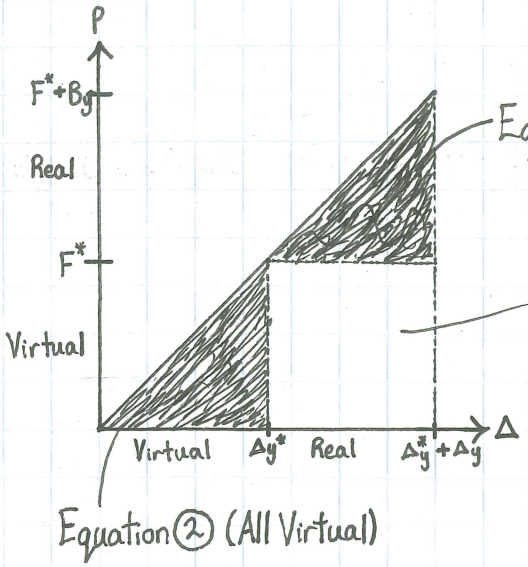
External = Internal

(3)

$$\frac{(B_x)(\Delta_x + \Delta_x^*)}{2} + \frac{(B_y + F^*)(\Delta_y + \Delta_y^*)}{2} = \frac{(F_{AB} + F^*_{AB})(\Delta_{AB} + \Delta^*_{AB})}{2} + \frac{(F_{BC} + F^*_{BC})(\Delta_{BC} + \Delta^*_{BC})}{2}$$

③ - ② - ①

Do it geographically



Equation ① (All Real)

Virtual Work = (Real Δ) · (Virtual Force)

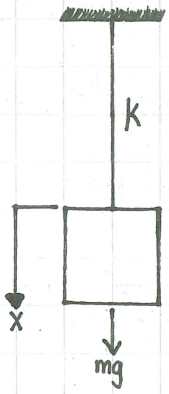
Note: Not divided by 2

Equation ② (All Virtual)

CIV102 - STRUCTURES and MATERIALS

Topic: Dynamic Amplification Factor (DAF) and Resonance

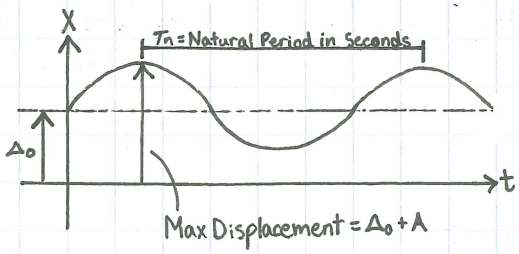
1) Case With No Damping



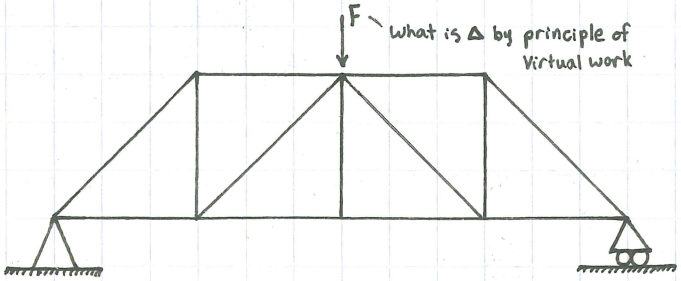
$$m\frac{d^2x}{dt^2} + kx = mg$$

$$x(t) = A\sin(\omega_n t + \phi) + \Delta_o$$

ω_n = natural frequency in $\frac{\text{radians}}{\text{second}}$



Static Displacement = $\Delta_o = \frac{mg}{k}$



For 1 Point Load $\rightarrow f_n \simeq \frac{15.76}{\sqrt{\Delta_o}}$ if Δ_o in mm

For Members With Uniform Loads $\rightarrow f_n = \frac{17.76}{\sqrt{\Delta_o}}$

2) Vibration With Damping

Damping means vibrational energy is statically absorbed

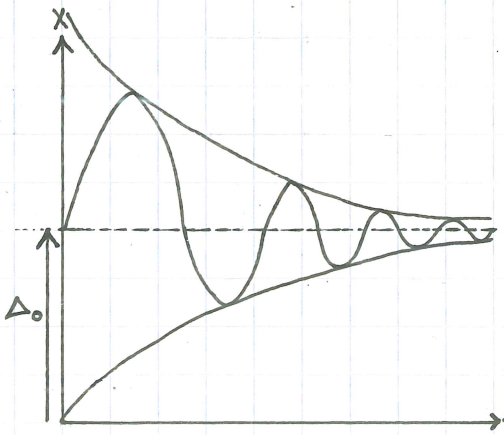
- Concept called critical damping $\rightarrow$ PHY293
- We define damping as β = fractional critical damping $\begin{cases} 0.01 = \beta \\ 0.05 = \beta \end{cases}$

Differential Equation

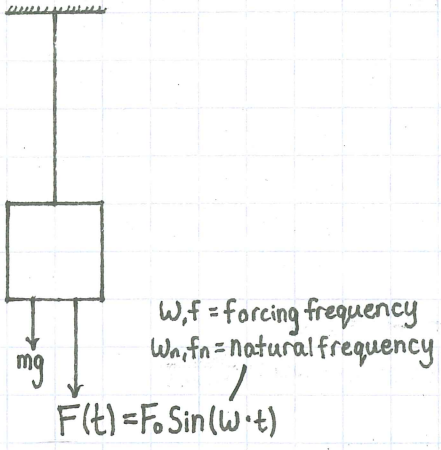
$$m\frac{d^2x}{dt^2} + C\underbrace{\frac{dx}{dt}}_{\text{damping}} + kx = mg$$

$$C = 2\beta\sqrt{m \cdot k}$$

$$x(t) = \underbrace{Ae^{-\beta \cdot \omega_n \cdot t}}_{\text{Exponential Decay}} \sin\left(\omega_n \underbrace{\sqrt{1-\beta^2}}_{\substack{\beta = \text{Small} \\ \beta^2 = \text{Really Small}}} t + \phi\right) + \Delta_o$$



3) Forced Vibration with Damping



Differential Equation

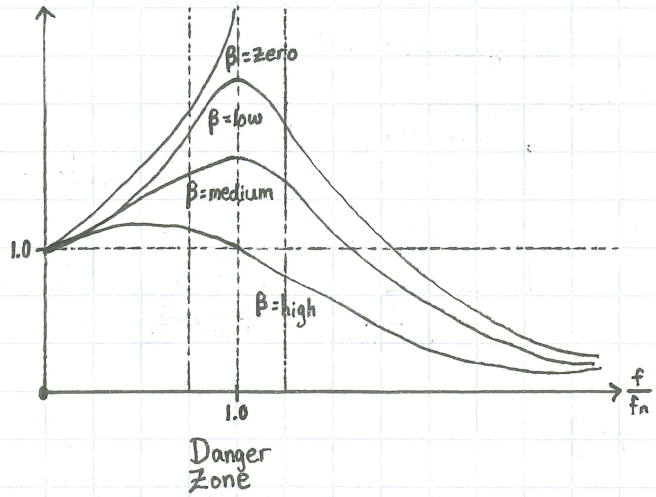
$$m\frac{d^2x}{dt^2} + 2\beta\sqrt{mk}\frac{dx}{dt} + kx = F_0 \sin(\omega t) + mg$$

$$X(t) = DAF \cdot \frac{F_0}{K} \cdot \sin(\omega t + \phi) + \Delta_0$$

DAF: dynamic amplification factor; ω: forcing frequency; Δ_0: static displacement

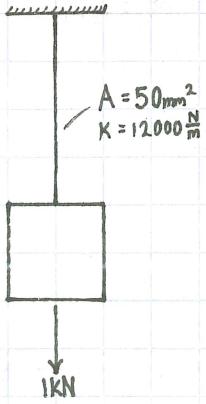
$$DAF = \frac{1}{\sqrt{\left(1-\left(\frac{f}{f_n}\right)^2\right)^2 + \left(\frac{2\beta f}{f_n}\right)^2}} \quad \text{(unitless)}$$

Plot DAF



Maximum force experienced by structure
Apparent $F_{max} = DAF \cdot F_0 + mg$

4) Example



Statics

$$\sigma = \frac{F}{A} = \frac{1000N}{50mm^2} = 20\,MPa$$

$$\Delta = \frac{F}{K} = \frac{1000N}{12000\frac{N}{m}} = 0.083m = 83.3mm$$

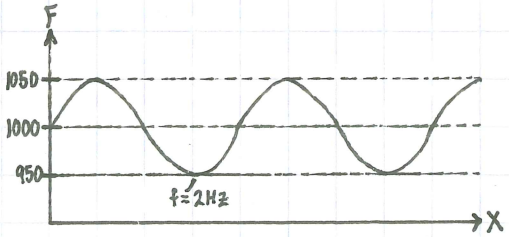
$$f_n = \frac{15.76}{\sqrt{\Delta}} = 1.727\,Hz$$

Dynamics

Tells You → Dynamic Loading = 1000N ± 50N @ 2Hz

DAF = 2.78

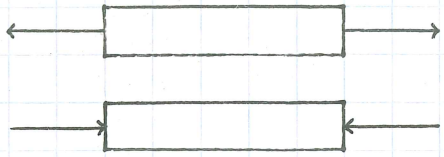
$$F_{max} = 1000N + DAF \cdot F_0$$
$$= 1000N + 2.78 \cdot 50$$
$$= 1139N \rightsquigarrow \sigma = 22.8\,MPa$$



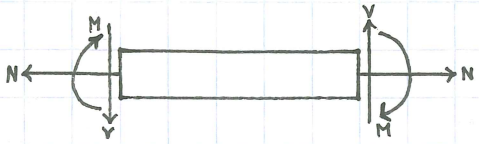
CIV102 - STRUCTURES and MATERIALS

Topic: Bending Moment Diagram/Shear Force Diagram (BMD/SFD)

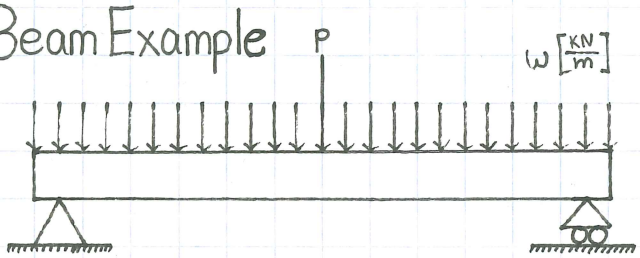
1) Truss Members



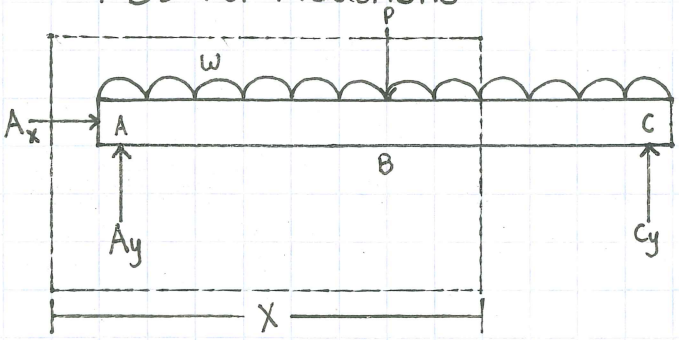
Beam/Column



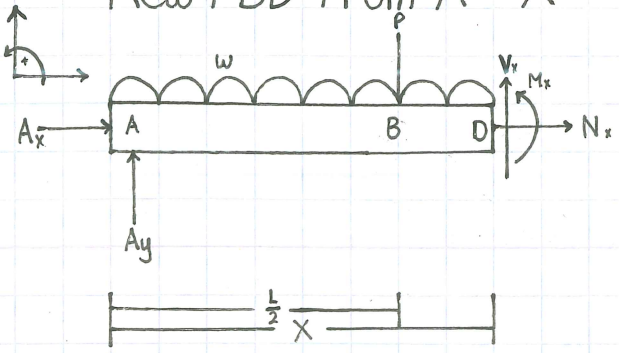
Beam Example

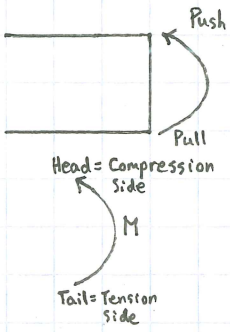


FBD For Reactions



New FBD From A → X





For any $\frac{L}{2} \le X \le L$

$\Sigma F_x = 0$, $A_x = -N_x$ (both = Zero)

$\Sigma F_y = 0$

$0 = A_y - P + V_x - wX$

$V_x = P + wX - A_y$

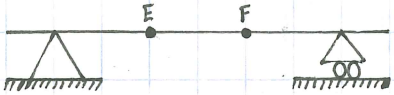
$\Sigma M_0 = 0$

$0 = M_x - A_y \cdot X + wX \cdot \frac{X}{2} + P(X - \frac{L}{2})$

$V_x = P + wX - A_y$

$M_x = -P(X - \frac{L}{2}) - \frac{wX^2}{2} + A_y \cdot X$

$M_x = \int V_x \, dx$ = the change in moment between E and F in a beam equals area under SFD from E to F

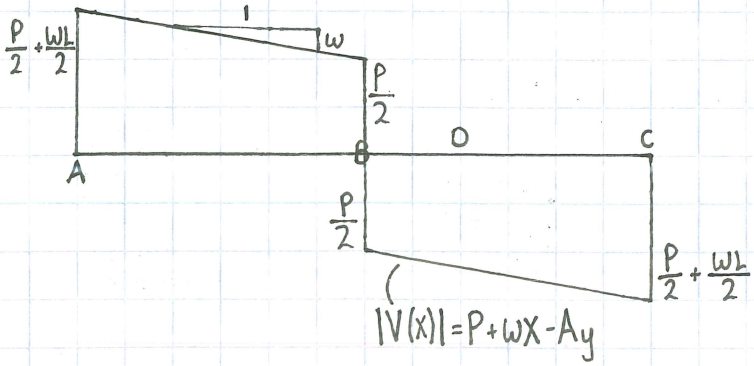


$V_x = \frac{dM}{dx}$ = Shear Force = Slope of BMD

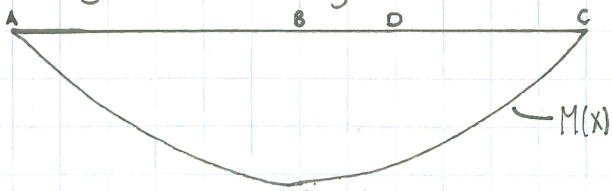
$w = \frac{dV}{dx}$ = Uniform Load = Slope of SFD

Draw

Shear Force Diagram



Bending Moment Diagram



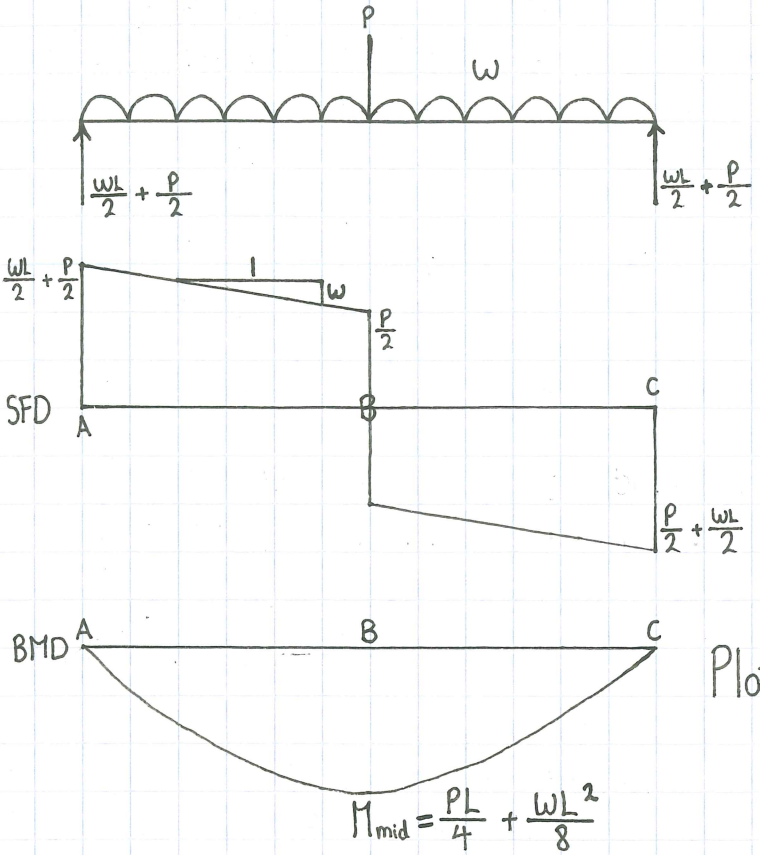
$M(X) = -P(X - \frac{L}{2}) - \frac{wX^2}{2} + A_y - X$

Max at Middle = Area of SFD from A to B = $\frac{P}{2}\frac{L}{2} + \frac{WL}{2}\frac{L}{2}\frac{1}{2}$

How to generate BMD, SFD without algebra

- Get reaction forces

For SFD - Accumulate Up + Down Forces

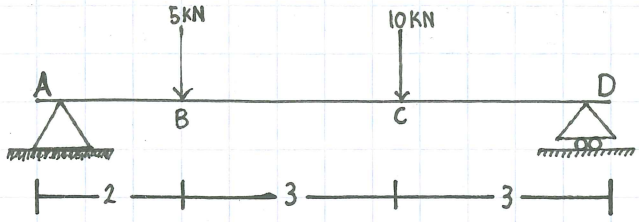


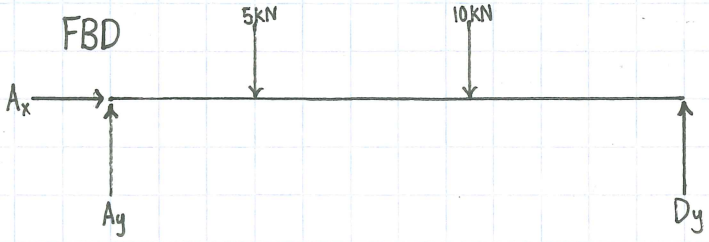
Plot on tension side

CIV102 - STRUCTURES and MATERIALS

V is slope of BMD
Change in Moment = Area Under SFD

1)





$\Sigma F_x = 0$, $A_x = 0$

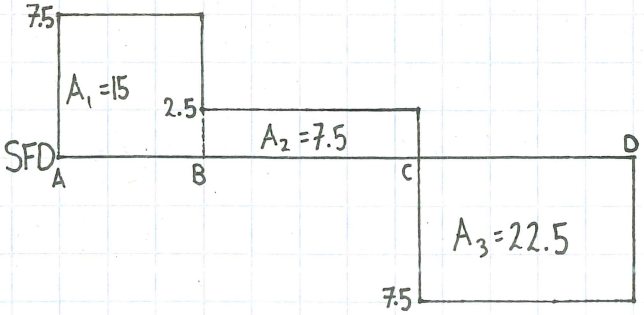
$\Sigma M_A = 0 = -5 \cdot 2 - 10 \cdot 5 + D_y \cdot 8$

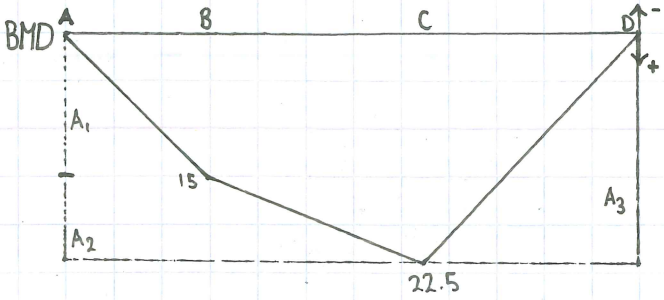
$D_y = 7.5\text{kN}$

$\Sigma F_y = 0$

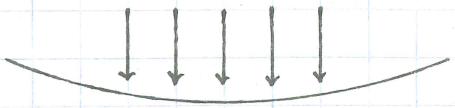
$0 = -5 - 10 + A_y + D_y$

$A_y = 7.5\text{kN}$

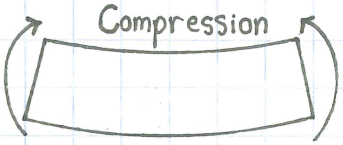




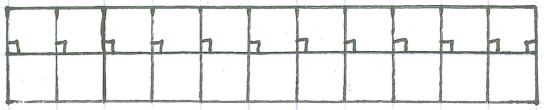
Draw on tension side

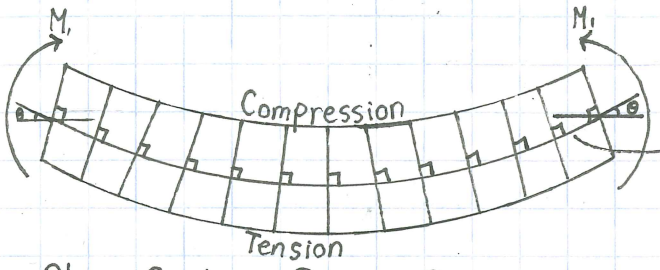


= Positive Bending
Positive Moments



2) Calculate Bending Stresses





$\varepsilon = 0$ at Neutral Axis = Centroid of X-Section

Plane Sections Remain Plane

$$\phi = \frac{d\theta}{dx} = \text{Curvature}$$

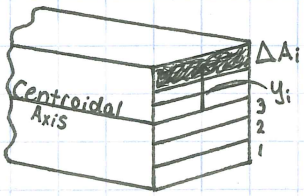
$$= \frac{\theta_2 - \theta_1}{L}$$

$$\varepsilon(y) = \phi y$$

$$M = EI\phi$$

$$I = \int y^2 dA = \frac{bh^3}{12}$$

3) $M \longleftrightarrow \sigma$



Strain in layer i

$$\varepsilon = \phi \cdot y \cdot i$$
$$\sigma = \varepsilon \cdot E$$
$$\sigma = \phi E y_i$$

Force in layer i

$$\Delta F_i = \phi E y_i \Delta A_i$$

Axial total force $= \sum_{i=1}^{n} \phi E y_i \Delta A_i$

Take limit as $\Delta A_i \to 0$ $\boxed{N = \phi E \int y\, dA}$

We want moment effects only

$$0 = E\phi \int y\, dA$$

$$\boxed{0 = \int y\, dA} \to \underline{\text{Use tomorrow}}$$

$$\text{Moment} = \sum_{i=1}^{n} \Delta F_i \cdot \text{lever arm}$$

$$= \sum_{i=1}^{n} \Delta F_i \cdot y_i$$

$$M = \sum_{i=1}^{n} \overbrace{\phi E y_i \cdot \Delta A_i}^{\Delta F_i} \cdot y_i$$

$$M = \phi E \sum_{i=1}^{n} y_i^2 \cdot \Delta A_i$$

Take $\lim \Delta A_i \to 0$

$$M = E\phi \int y^2 \, dA$$

$$M = EI\phi$$

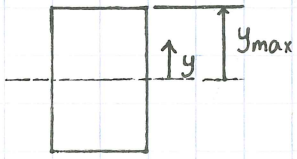
Above we said $\sigma = \phi E y$

$$\phi = \frac{\sigma}{Ey}$$

Sub in

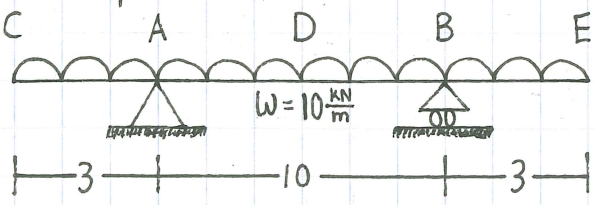
$$M = EI \cdot \frac{\sigma}{Ey}$$

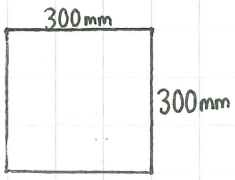
$$\boxed{\sigma = \frac{My}{I}}$$ Navier's Equation



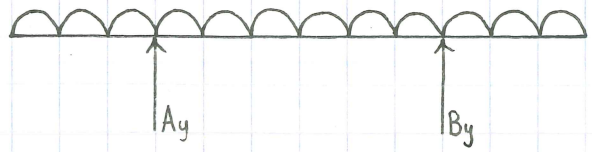
$$\boxed{\sigma_{max} = \frac{M_{max} y_{max}}{I}}$$

4) Example



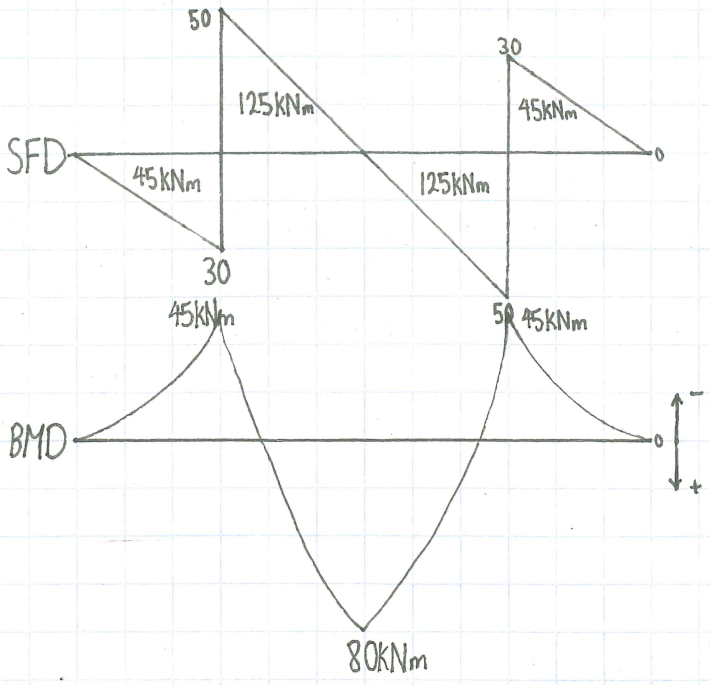


$\sigma_{max} = ?$



Total Force Down = 10·16 = 160kN

Symmetry $A_y = B_y = 80\text{kN}$



$$\sigma = \frac{My}{I} \qquad I = \frac{bh^3}{12} = 675 \times 10^6 \, mm^4$$

$$\sigma_{max} = \frac{M_{max} \cdot y_{max}}{I} \qquad y_{max} = 150 = \frac{h}{2}$$

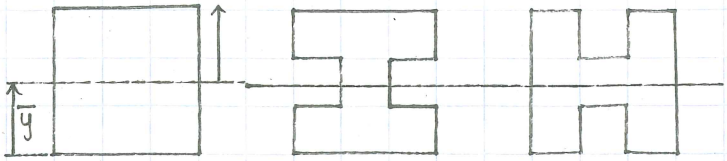
$$= \frac{80 \times 10^6 \, Nmm \cdot 150 \, mm}{675 \times 10^6 \, mm^4} \qquad M_{max} = 80 kNm$$

$$= 17.78 MPa$$

CIV102 - STRUCTURES and MATERIALS

Topic: $\bar{y}$ and I

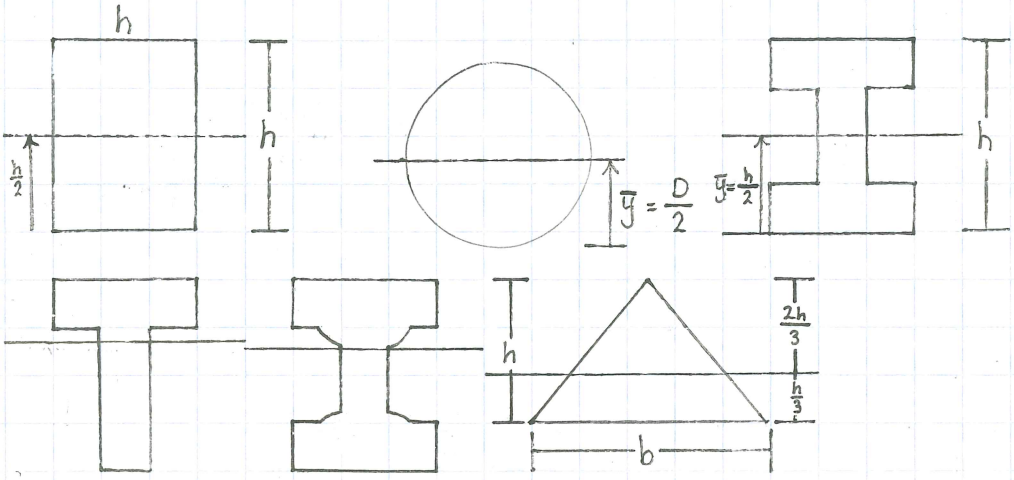
$\sigma = \frac{My}{I}$



1) Why I?

$\sigma = \frac{My}{I}$ $\quad M = EI\phi$

2) Centroids

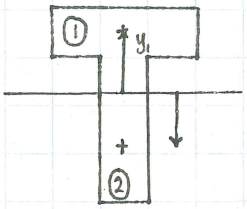


Last Class

$N = E\phi \int y\,dA$

If only M, N = 0

$0 = \int y\,dA$

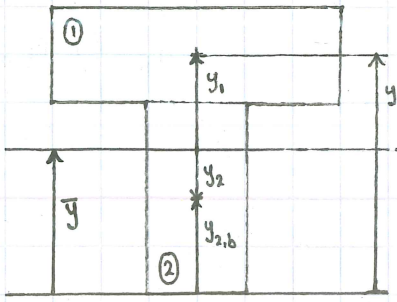


$0 = A_1 y_1 + A_2 y_2$

y_1 = distance from centroid of A_1 to centroid of entire section

A_1, A_2 = Positive

y_1, y_2 = One Positive, One Negative



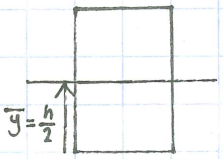
$$0 = A_1 \cdot y_1 + A_2 \cdot y_2$$
$$= A_1(y_{1,b} - \bar{y}) + A_2(y_{2,b} - \bar{y})$$
$$= \sum_{i=1}^{n} A_1(y_{1,b} - \bar{y})$$
$$0 = \sum_{i=1}^{n} A_i y_{i,b} - \sum_{i=1}^{n} A_i \cdot \bar{y}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} A_i \cdot y_{i,b}}{\sum_{i=1}^{n} A_i} = \frac{\sum_{i=1}^{n} A_i \cdot y_{i,b}}{\text{Total Area}}$$

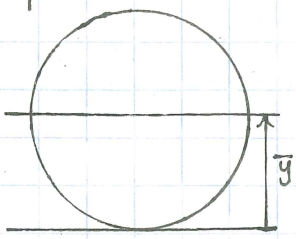
3) Calculate I

$$I = \int y^2 \, dA$$

I for some basic shapes

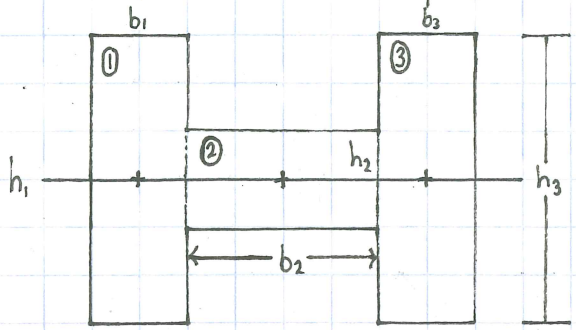


$$I = \frac{bh^3}{12}$$

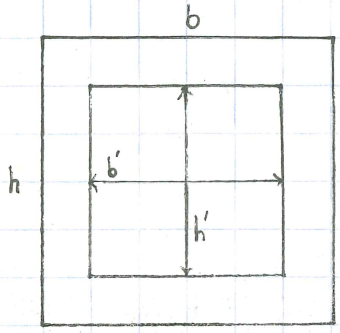


$$I = \frac{\pi D^4}{64}$$

a) Cases where all subparts have same centroidal axis height

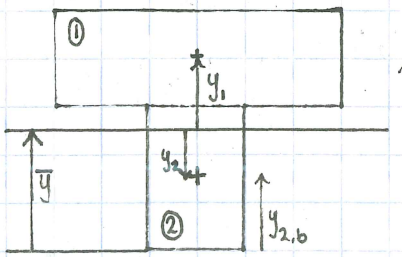


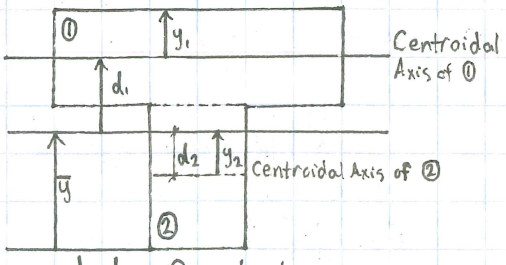
$$I_{total} = I_1 + I_2 + I_3$$
$$= \frac{b_1 h_1^3}{12} + \frac{b_2 h_2^3}{12} + \frac{b_3 h_3^3}{12}$$



$$I_{total} = I_{Solid} - I_{Hollow}$$
$$= \frac{bh^3}{12} - \frac{b'h'^3}{12}$$

b) I if centroidal axis do not align





d_1, d_2 = Constants

d_1 = distance from centroid of A_1 to Centroid of whole section

$$I = \int y^2 \, dA$$
$$= \int_{A_1} (y_1 + d_1)^2 \, dA + \int_{A_2} (y_2 + d_2)^2 \, dA$$
$$= \int_{A_1} y_1^2 \, dA + \int_{A_1} 2 y_1 d_1 \, dA + \int_{A_1} d^2 \, dA + \int y_2^2 \, dA + \int 2 y_2 d_2 \, dA + \int d_2^2 \, dA$$
$$= I_o + 2d_1 \cancel{\int y_1 \, dA}^{0} + d_1^2 \int dA + I_{o2} + 2d_2 \cancel{\int y_2 \, dA}^{0} + d_2^2 \int dA$$

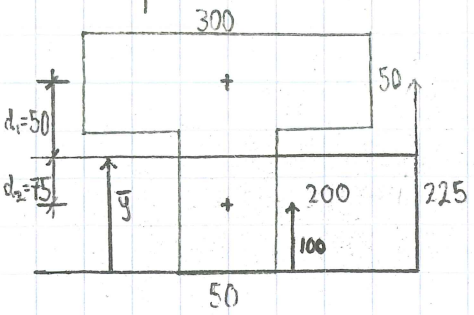
I_o: I of area 1 about its own centroid $= \frac{bh^3}{12}$

$\int y \, dA = 0$ as

$$\therefore I = I_{o1} + I_{o2} + A_1 d_1^2 + A_2 d_2^2$$

Parallel Axis Theorem

Example



$$\bar{y} = \frac{A_1 y_{1,b} + A_2 \cdot y_{2,b}}{A_{Total}}$$

$$= \frac{300 \cdot 50 \cdot 225 + 200 \cdot 50 \cdot 100}{300 \cdot 50 + 200 \cdot 50}$$

$$= 175\,mm$$

$$I = \frac{b_1 h_1^3}{12} + \frac{b_2 h_2^3}{12} + A_1 d_1^2 + A_2 d_2^2$$

$$\frac{b_1 h_1^3}{12} = 3.125 \times 10^6\,mm^4$$

$$\frac{b_2 h_2^3}{12} = 33.3 \times 10^6\,mm^4$$

$$A_1 d_1^2 = 37.5 \times 10^6\,mm^4$$

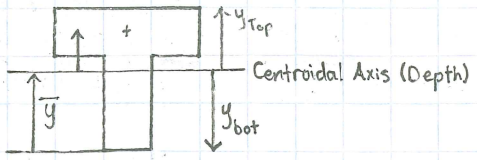
$$A_2 d_2^2 = 56.25 \times 10^6\,mm^4$$

$$\sum I_{Total} = 130.2 \times 10^6\,mm^4$$

CIV102 - STRUCTURES and MATERIALS

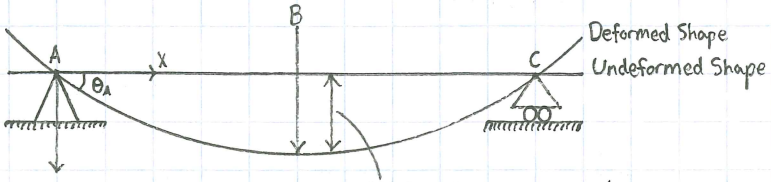
Topic: θ + Δ

1) $\sigma = \frac{My}{I}$



$|y_{bot}| > |y_{top}|$

2) Displacements of Beams



$y(x)$ = displacement

Small Displacements

θ_A = Angle = Shape = $\frac{dy}{dx} = \tan\theta$

$\Phi \longrightarrow M = EI\Phi$

$\Phi = \frac{M}{EI}$

$\Phi = \frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

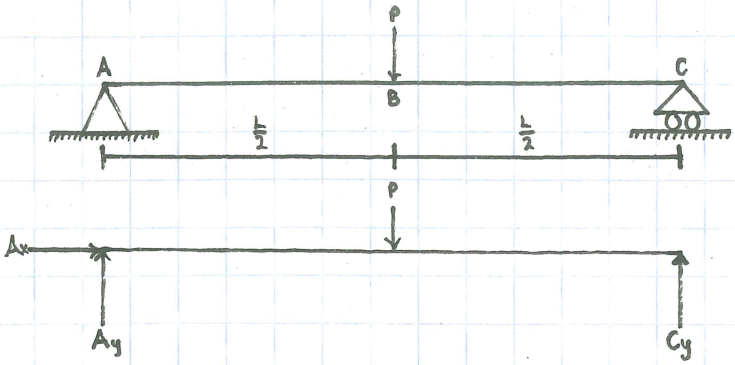
$\boxed{\Phi = \frac{d^2y}{dx^2}}$ if angles are small

For larger angles

$$\Phi = \frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$

3) ϕ diagram

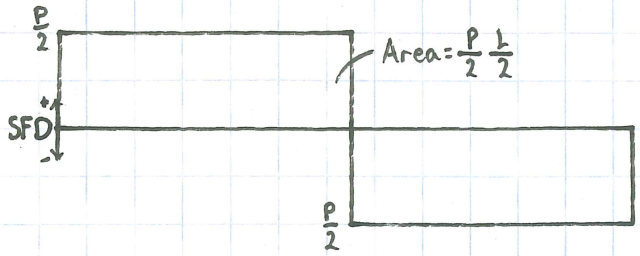
$\phi = \frac{M}{EI}$

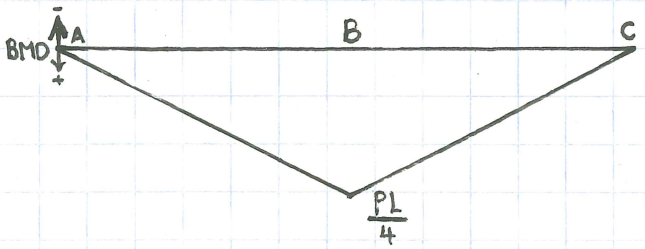


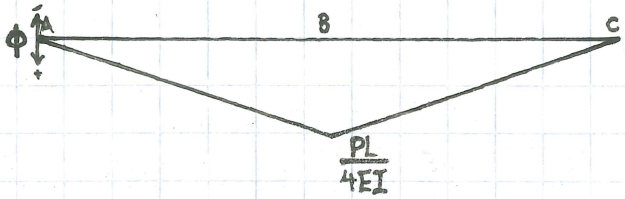
$\Sigma F_x = 0,\ A_x = 0$

$\Sigma M_A = 0 = -P \cdot \frac{L}{2} + C_y \cdot L,\ C_y = \frac{P}{2}$

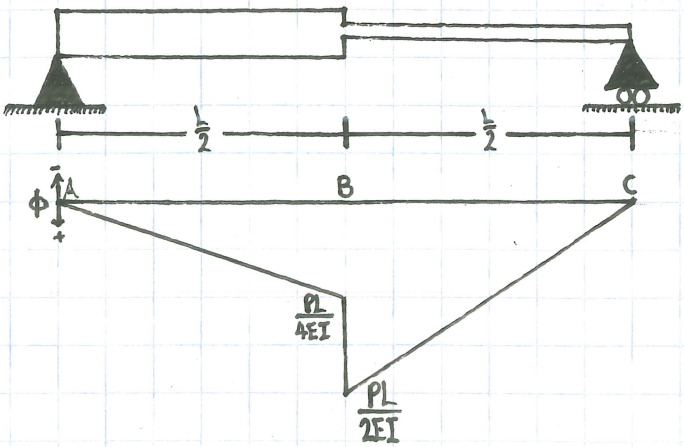
$\Sigma F_y = 0 = A_y + C_y - P,\ A_y = \frac{P}{2}$







Problem 2



4) Solve for Displacements and Angles

↳ Calculus
↳ Moment Area Theorems

A) MAT #1 ⟶ θ

$\phi = \frac{d^2y}{dx^2}$, Integrate Once

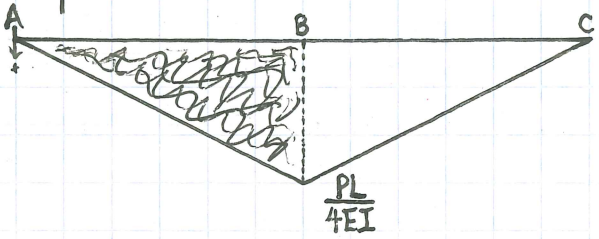
$\int \phi \, dx + C = \frac{dy}{dx} = \theta$

$\theta = \int \phi \, dx + C$

Moment Area Theorem #1

The change in slope between A and B in a beam is equal to the area under ϕ diagram between A and B.

Example



$\theta_A = ?$

by symmetry θ_B = Zero

$\theta_A - \theta_B$ = Shaded Area $= \frac{1}{2} \cdot \frac{PL}{4EI} \cdot \frac{1}{2}$

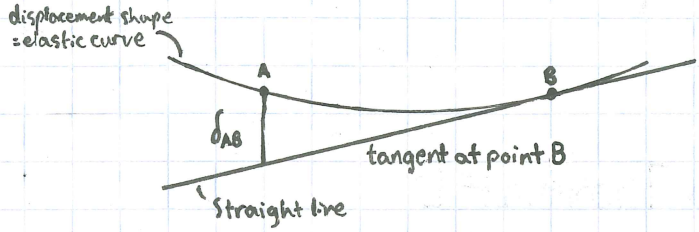
$$\boxed{\theta_A = \frac{PL^2}{16EI}}$$

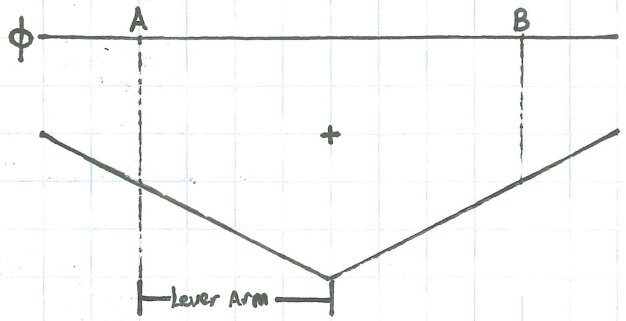
B) MAT # 2

$$\int \Phi \, dx + C_1 = \frac{dy}{dx}$$

$$\underbrace{\iint \Phi \, dx}_{\text{First Moment of Area}} + \underbrace{C_1 x + C_2}_{\text{Equation of a Line}} = y$$

y: displacement perpendicular to undeformed shape from the straight line





Moment Area Theorem #2

The tangental deviation δ_{AB} at point A between the elastic curve and the tangent drawn from B is equal to the first moment of area of the Φ diagram between A and B with moments taken about point A.

δ_{AB}

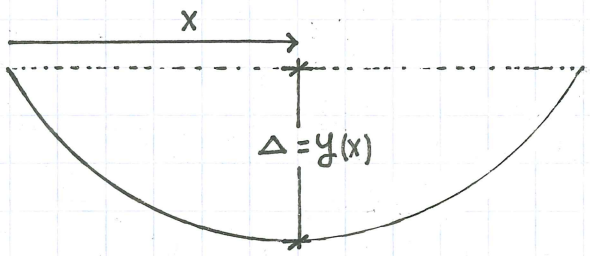
A → Where to get δ
B → Where is the tangent
A → Where to get lever arm

$\delta_{AB} = A_B \cdot (\text{lever arm})$

CIV102 - STRUCTURES and MATERIALS

Topic: MAT + Δ + θ

1) $y(x)$	Δ	Displacement	$\int \theta\, dx$	Deformations
$\frac{dy}{dx}$	θ	Slope	$\int \frac{M}{EI}\, dx$	
$\frac{d^2y}{dx^2}$	ϕ	Curvature	$\frac{M}{EI}$	
$EI\frac{d^2y}{dx^2}$	M	Moment	$\int V\, dx$	* EI
$EI\frac{d^3y}{dx^3}$	V	Shear	$\int w\, dx$	Forces
$EI\frac{d^4y}{dx^4}$	w	Distributed Load		

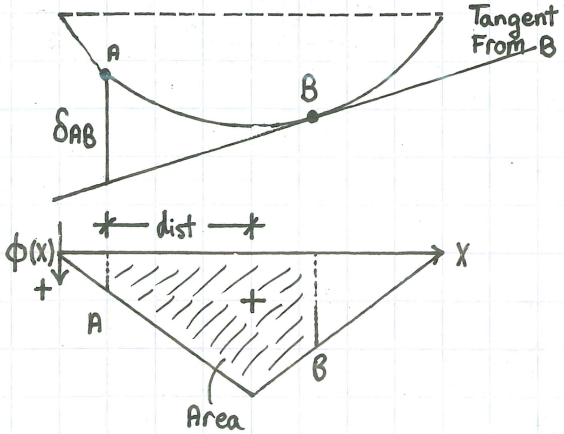


$$\Delta = \iiiint \frac{w}{EI}\, dx$$

2) Moment Area Theorems (MAT)

MAT #1 $\longrightarrow \Delta\theta_{AB} = |\theta_B - \theta_A| = \int_A^B \phi(x)\, dx$ = Area under ϕ diagram between A and B

MAT #2 $\longrightarrow \delta_{DT} = \underbrace{\bar{X}_{DT}}_{\text{Distance}} \underbrace{\int_D^T \phi(x)\, dx}_{\text{Area}}$ = Distance between the deflected beam at D and the line that is tangent to T equals the <u>area</u> under the ϕ diagram times the <u>distance</u> from the centroid of that diagram to D



δ_{AB} = Area · dist
A - Where is δ
B - Where is Tangent
A - Where is Distance

Note: $\delta_{AB} \neq \delta_{BA}$

Procedure to Find θ and Δ of Beams

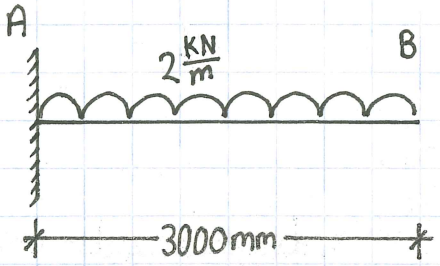
1) Calculate the reaction forces, SFD, BMD, Φ Diagram
2) Sketch the displaced shape
   - ↳ The beam can't move up/down at support
   - ↳ The BMD tells you which way it is curving
3) Find places where the Δ or θ are known
   - ↳ Supports
   - ↳ Loading Conditions
4) Use MAT #1 to find θ
5) Use MAT #2 to find Δ

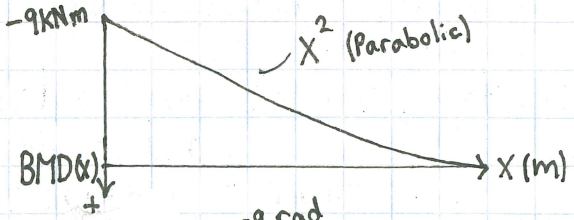
Senario 1 - Fixed Supports

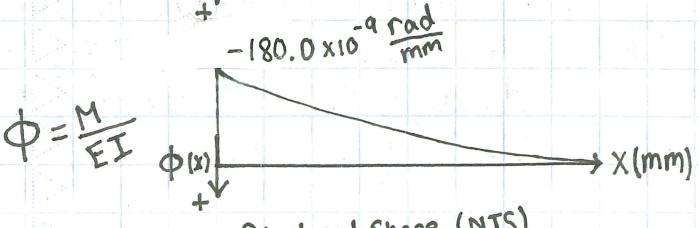
At fixed support, $\theta = 0$

Example Find θ_B and ΔB

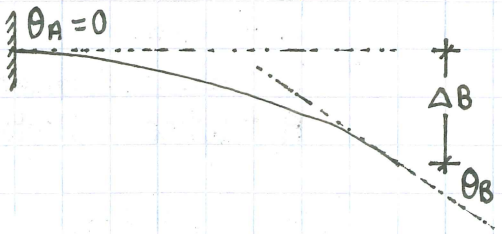
EI Constant = $5 \times 10^{13}\ Nmm^2$







Displaced Shape (NTS)



$\Delta\theta_{AB} = |\theta_B - \cancel{\theta_A}^{0}|$

θ_B = Area under Φ

$= \frac{1}{3}bh$

$= \left(\frac{1}{3}\right)(3000mm)\left(180.0 \times 10^{-9}\ \frac{rad}{mm}\right)$

$\boxed{\theta_B = 180.0 \times 10^{-6}\ rad}$

$\delta_{BA} = \Delta_B = (180.0 \times 10^{-6})\left(\frac{3}{4}\right)(3000mm)$

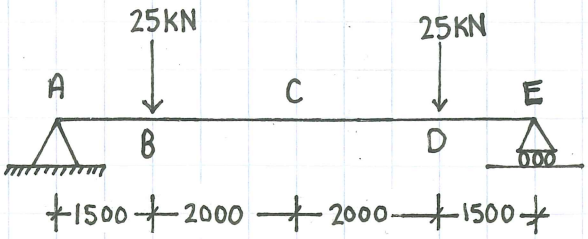
$\boxed{\Delta B = 0.405mm}$

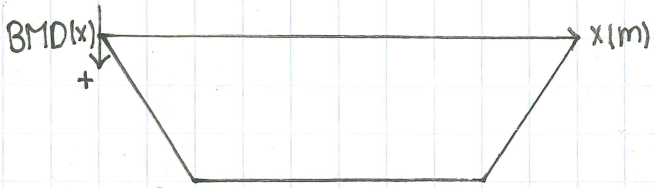
Senario 2 - Symmetry

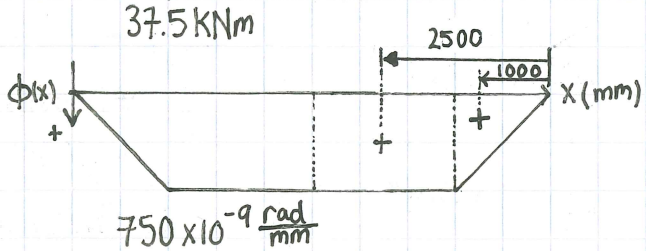
If the geometry and loading are symmetrical, then max deflection location is known

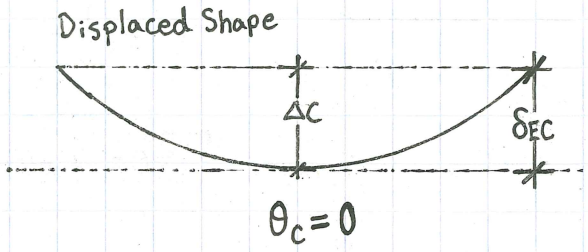
↳ At the location of max Δ, $\theta = 0$

Example Find ΔC $EI = 5 \times 10^{13}\ Nmm^2$









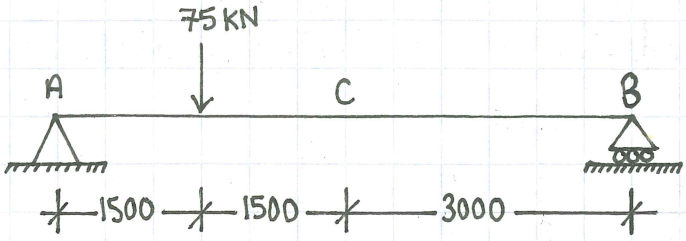
$$\delta_{EC} = \Delta C = (2000\,mm)\left(750 \times 10^{-9}\ \frac{rad}{mm}\right)(2500\,mm) + \left(\frac{1}{2}\right)(1500\,mm)\left(750 \times 10^{-9}\ \frac{rad}{mm}\right)(1000\,mm)$$

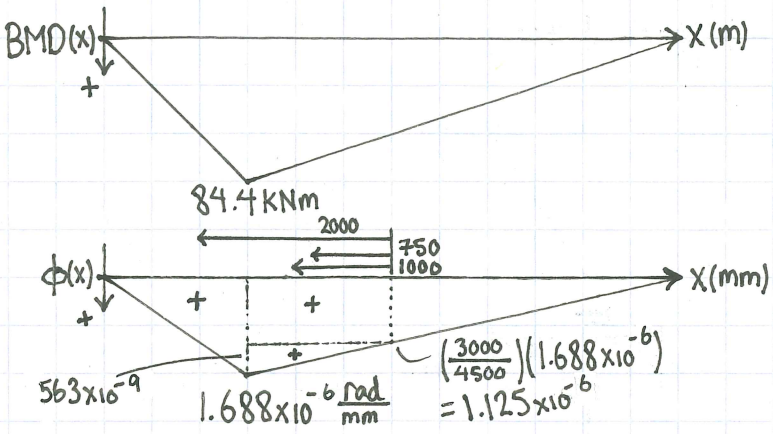
$$\boxed{\Delta C = 4.31\,mm}$$

Scenario 3 – Everything Else

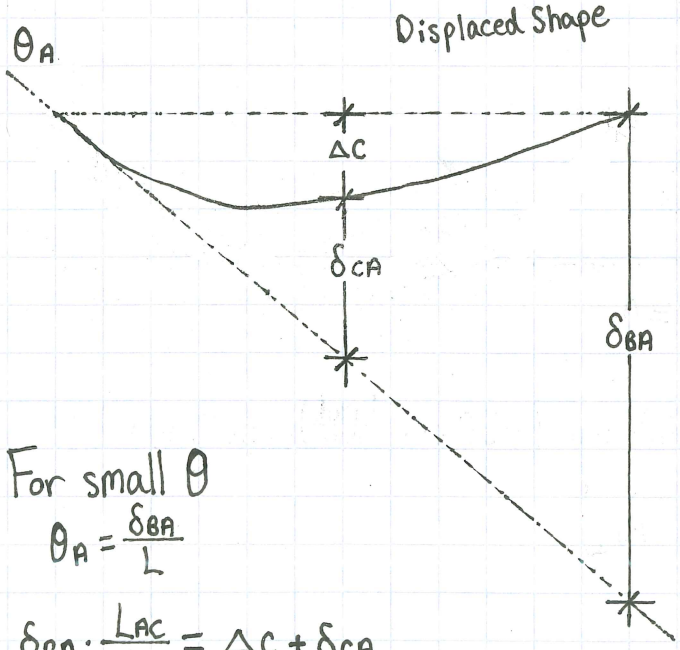
No horizontal tangents are known Use similar triangles

Example Find Δ_C $EI = 5 \times 10^{13}\ Nmm^2$





Displaced Shape



For small θ

$\theta_A = \frac{\delta_{BA}}{L}$

$\delta_{BA} \cdot \frac{L_{AC}}{L_{AB}} = \Delta C + \delta_{CA}$

$\frac{1}{2}\delta_{BA} = \Delta C + \delta_{CA}$

$\delta_{BA} = 17.72\ mm$

$\delta_{CA} = (1500)(1.688 \times 10^{-6})(\frac{1}{2})(2000)$
$+ (1500)(1.125 \times 10^{-6})(750)$
$+ (1500)(563 \times 10^{-9})(\frac{1}{2})(1000)$

$\delta_{CA} = 4.22\ mm$

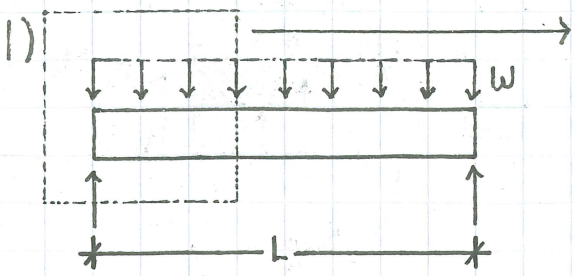
$\Delta C = \frac{1}{2}\delta_{BA} - \delta_{CA}$
$= (\frac{1}{2})(17.72) - 4.22$

$\boxed{\Delta C = 4.64\ mm}$

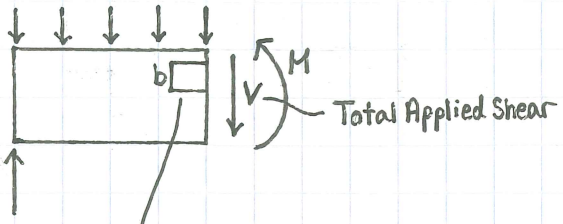
CIV102 - STRUCTURES and MATERIALS

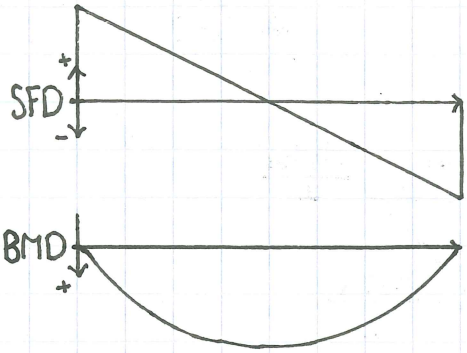
Topic: Shear Stress

1)

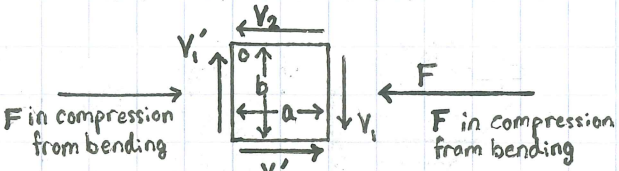


F.B.D.





F.B.D. of Chunk of Beam



$\int \frac{My}{I} dA$

V_1 = Part of Total Shear V

$\sum F_y = 0 \Rightarrow V_1' = V_1$

$\sum F_x = 0 \Rightarrow V_2 = V_2'$

And couple from V_2 will cancel out Force Couple from V_1

$\sum M_o = 0$

$0 = -V_1 \cdot a + V_2 \cdot b$

$V_1 \cdot a = V_2 \cdot b$

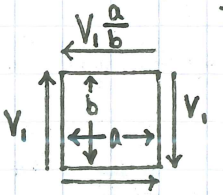
$V_2 = V_1 \frac{a}{b}$

2) Define Shear Stress

τ = Shear Stress = $\frac{\text{Force}}{\text{Area}}$ = MPa

Force is parallel to Area

t = thickness

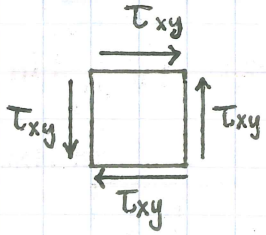


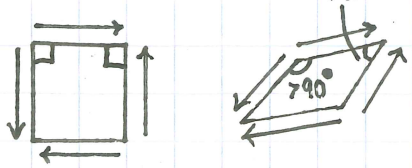
$\tau_{\text{on sides}} = \frac{\text{Force}}{\text{Area}} = \frac{V_1}{b \cdot t}$

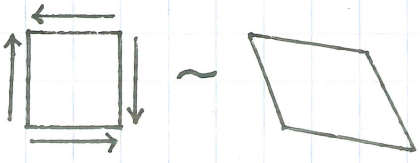
$\tau_{\text{on top}} = \frac{V_1 a}{b a t} = \frac{V_1}{bt}$

Shear stress on top + bottom + left + right sides are all the same

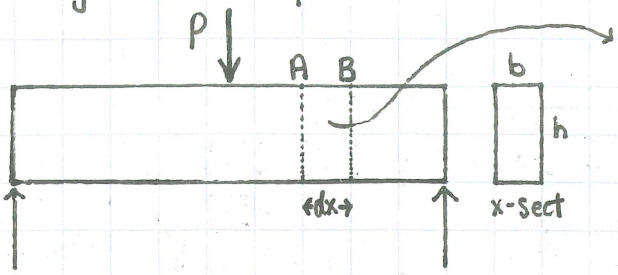
$\tau_{xy} = V_{xy}$

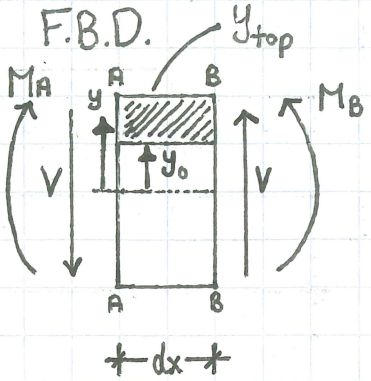






3) τ_{xy} Versus Depth

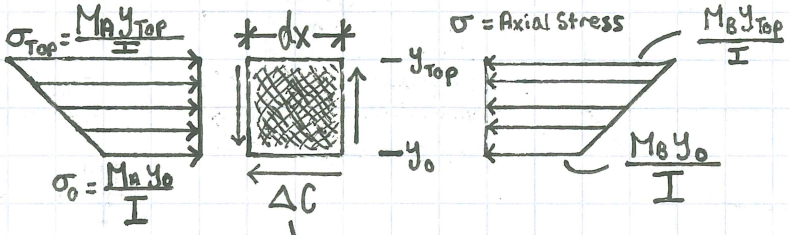




$$\Sigma M_A = 0$$
$$0 = M_B - M_A + V \cdot dx$$
$$M_A = M_B + V \cdot dx$$

$$\sigma_x = \frac{My}{I}$$

F.B.D. of Shaded Region



Caused by $M_A \neq M_B$
Induces τ_{xy} Stresses

Calculate ΔC

$$\text{Force on Left} = \int_{y_o}^{y_{top}} \sigma \, dA = \int_{y_o}^{y_{top}} \frac{M_A y}{I} \, dA$$

$$\text{Force on Right} = \int_{y_o}^{y_{top}} \frac{M_B y}{I} \, dA$$

$$\Delta C = \frac{(M_B - M_A)}{I} \int_{y_o}^{y_{top}} y \, dA$$

Value of $(M_B - M_A)$

$$(M_B - M_A) = M_B - (M_B + V\,dx)$$
$$= -V\,dx$$

Take absolute value

$$(M_B - M_A) = V\,dx$$

1855 Jourawski (Zurauski)

$$\tau_{xy} \text{ @Depth } y_o = \frac{\Delta C}{dx \cdot b}$$

Width of cross section @ depth y_o

$$\tau_{xy} = \frac{V\,dx}{I} \int_{y_o}^{y_{top}} y \, dA \cdot \frac{1}{dx \cdot b}$$

$$\tau_{xy} = \frac{V}{Ib} \int_{y_o}^{y_{top}} y \, dA$$

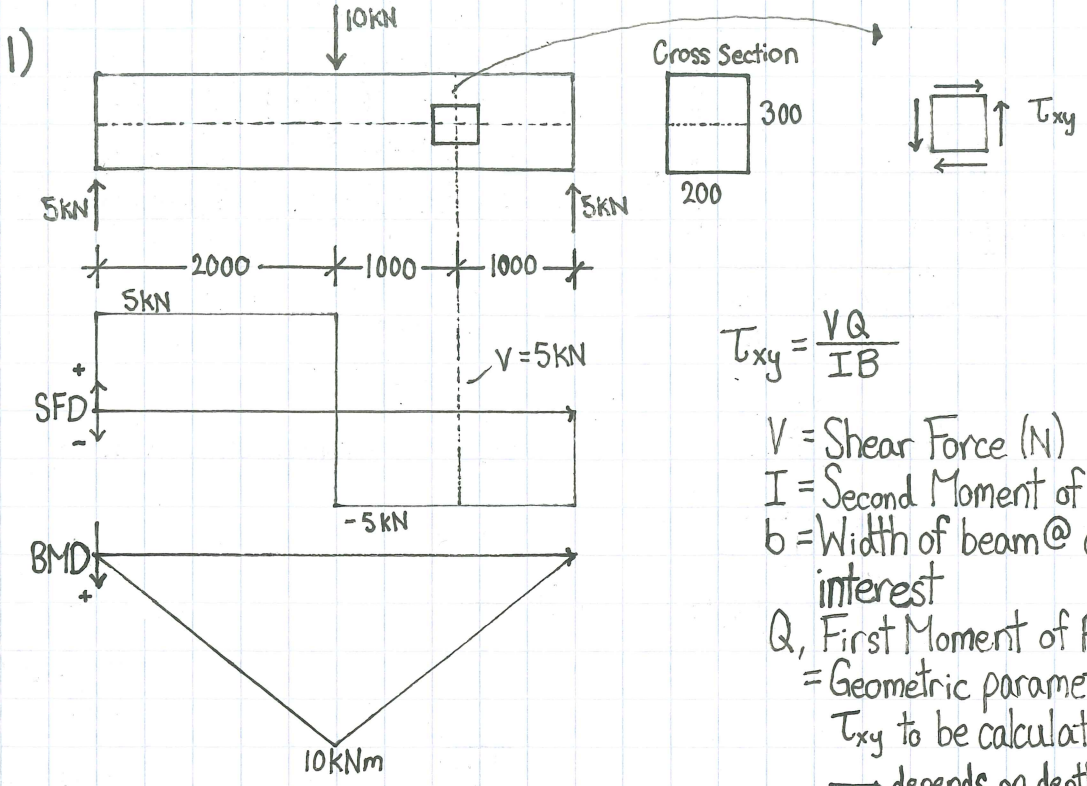
1st Moment of Area Q

$$\boxed{\tau_{xy} = \frac{VQ}{Ib}}$$

CIV102 - STRUCTURES and MATERIALS

Topic: τ_{xy} & $\frac{VQ}{IB}$

1)



$$\tau_{xy} = \frac{VQ}{IB}$$

V = Shear Force (N)

I = Second Moment of Area (mm^4)

b = Width of beam @ depth of interest

Q, First Moment of Area
= Geometric parameter to allow τ_{xy} to be calculated
⟶ depends on depth of interest

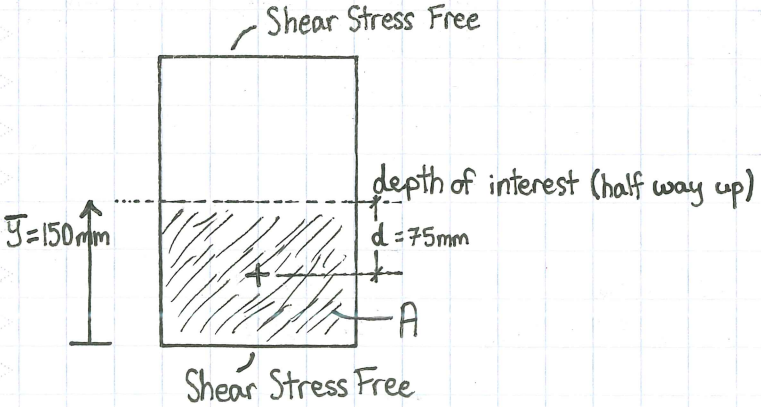
$Q = A \cdot d$

A = area between point of interest and a shear stress free surface

d = distance from the centroid of the area of interest to the centroid of the entire cross section

1st moment of area between a shear stress free surface (usually top or bottom) and the depth of interest with lever arm, d, taken as distance from centroid of area to centroidal axis of whole cross section, $\bar{y}$

Example Case Calculate τ_{xy} @ Midheight



$$\tau_{xy} = \frac{VQ}{IB}$$

$$I = \frac{bh^3}{12} = 450 \times 10^6 \ mm^4$$

$V = 5000N \quad b = 200mm$

$Q = Ad = 200 \cdot 150 \cdot 75$
$= 2.25 \times 10^6 \ mm^3$

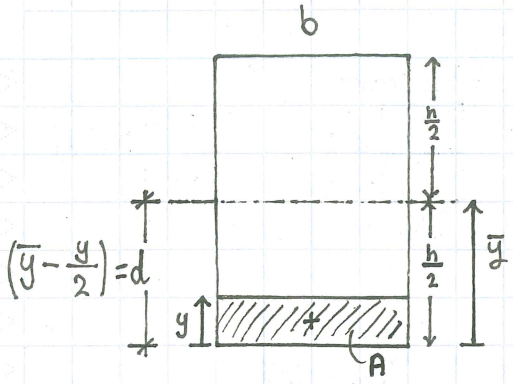
$$\tau_{xy@\bar{y}} = \frac{VQ}{IB} = \frac{5000 \cdot 2.25 \times 10^6}{450 \times 10^6 \cdot 200} = 0.125MPa$$

* Design for max τ_{xy} stress

Q is a maximum @ $\bar{y}$ ∴ $\tau_{xymax} = 0.125MPa$ here

$$\text{Average Shear Stress} = \frac{V}{bh} = \frac{5000}{200 \cdot 300} = 0.0833MPa < \tau_{xymax}$$

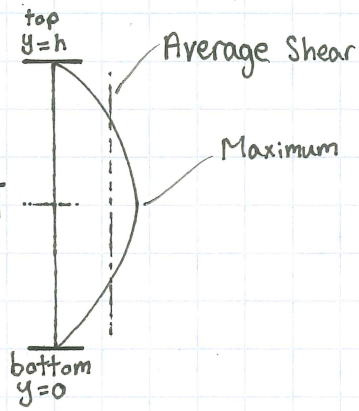
2) Variation of τ_{xy} vs. Depth



$$Q = A \cdot d$$
$$= by \cdot \left(\bar{y} - \frac{y}{2}\right)$$
$$= by \cdot \left(\frac{h}{2} - \frac{y}{2}\right)$$
$$= \frac{by}{2}(h - y) \longrightarrow \text{Parabolic with depth}$$

Plot τ_{xy}

$$\tau_{xy} = \frac{VQ}{IB}$$
$$= \frac{V \cdot by(h-y)/2}{\frac{2bh^3}{12} \cdot b}$$
$$= \text{Something!}$$

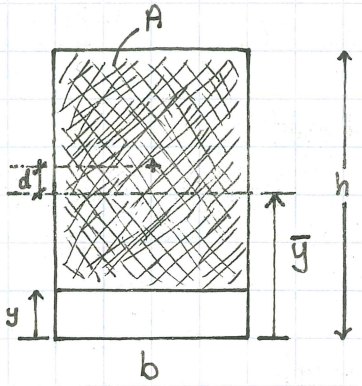


$$\tau_{xy} @ \frac{h}{2},\ y = \frac{h}{2}$$

$$\tau_{xy} = \frac{V \cdot h \cdot h/2}{2 \cdot 2 \cdot b h^3 / 12}$$

$$\tau_{xy} = \frac{3V}{2bh}$$

Redo with calculation of Q from top

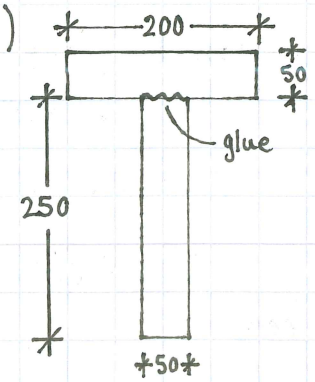


$$d = \frac{h-y}{2} + y - \bar{y}$$
$$= \frac{h}{2} - \frac{y}{2} + y - \frac{h}{2}$$
$$d = \frac{y}{2}$$

$$Q = A \cdot d$$
$$= b(h-y)\frac{y}{2}$$

Same as from bottom.

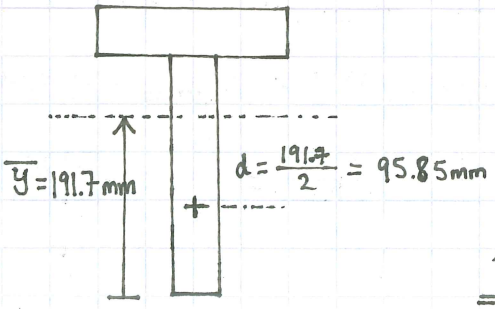
3)



i) $\tau_{xy\,max}$?
ii) τ_{xy} @ glue joint

$\bar{y} = 191.7\text{mm}$ $I = 192.2 \times 10^6\,\text{mm}^4$
$V = 40\,\text{kN}$

i) τ_{xy} will be max @ $\bar{y}$

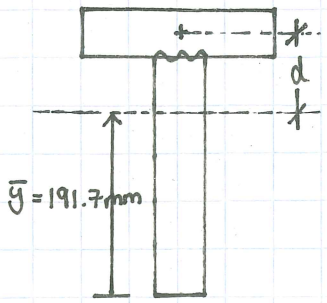


$$Q = A \cdot d$$
$$= 191.7 \cdot 50 \cdot \frac{191.7}{2}$$
$$= 918.7 \times 10^3\,\text{mm}^3$$

$$\tau_{xy} = \frac{40000 \cdot 918.7 \times 10^3}{192.2 \times 10^6 \cdot 50}$$

$$\tau_{xy\,max} = 3.82\text{MPa}$$

ii) τ_{xy} on glue joint



$$d = 250 - \bar{y} + 25 \Rightarrow d = 83.3\,mm$$

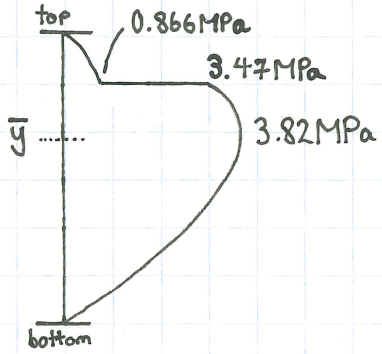
$$Q = 200 \cdot 50 \cdot 83.3 \Rightarrow Q = 833 \times 10^3\,mm^3$$

$$\tau_{xy,glue} = \frac{VQ}{IB}$$

$$= \frac{40000 \cdot 833 \times 10^3}{192.2 \times 10^6 \cdot 50}$$

$$\underline{\tau_{xy,glue} = 3.47\,MPa}$$

τ_{xy} Distribution



CIV102 - STRUCTURES and MATERIALS

Topic: Timber!

1) Timber → Engineered Wood Product

Advantages	Disadvantages
Light	Burns
Reasonably Strong	Splinters
Cheap	Rot
Easily Change Shape	Warp
	Higher variability in strength than other materials } Uncertainty

Isotropic → Steel
Anisotropic → Wood

Size effect
→ Allowable stress depends on size of member (larger = stronger)

2) Types of Wood

Hardwood → Oak
Softwood → Pine

Appendix D

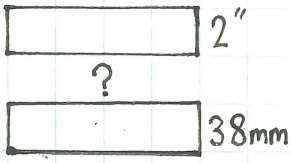
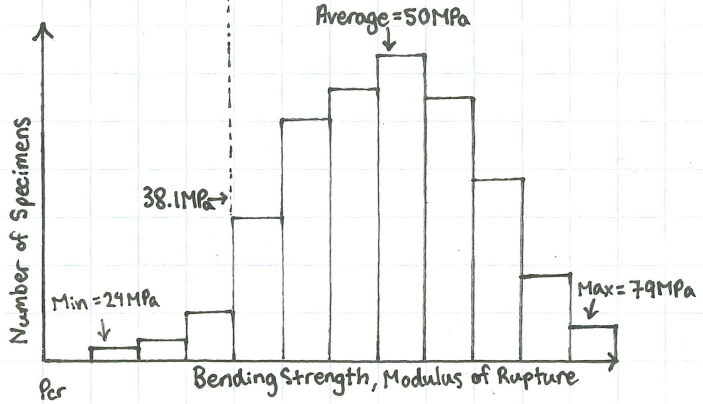
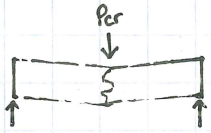


Table - D

5th percentile → Strength That is higher than 5% of all test values

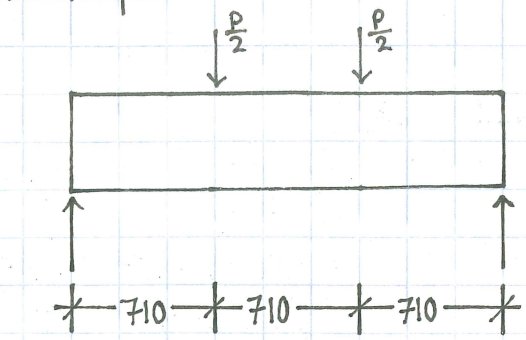


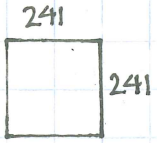


$\frac{My}{I}$ = Modulus of Rupture

Bending Stress
Shear Stress
} Divide by F.O.S. = 1.5 to set allowable stresses

3) Example

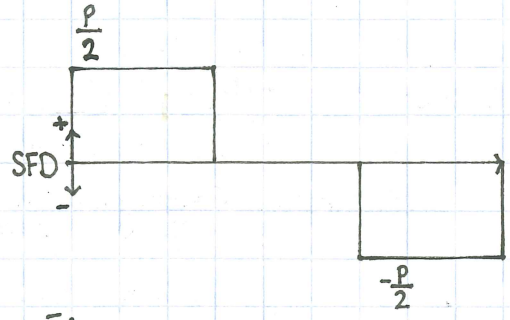


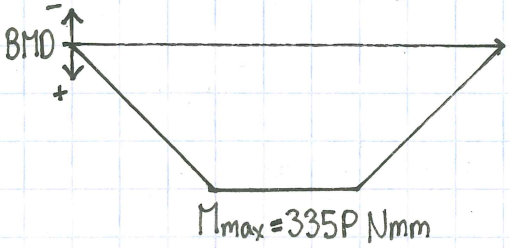


Douglas Fir
- Bending Strength = 24 MPa
- Shear Strength = 1.1 MPa

$$\text{Allowable Bending} = \frac{24}{1.5} = 16\,\text{MPa}$$

$$\text{Allowable } \tau_{xy} = \frac{1.1}{1.5} = 0.73\,\text{MPa}$$





What is max safe value of P?

Check Bending

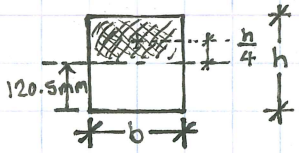
$$I = \frac{bh^3}{12} = 281 \times 10^6\,\text{mm}^4$$

$$\sigma_{\text{Allowable}} = \frac{My}{I}$$

$$16 = \frac{355P \cdot 241}{2 \cdot 281 \times 10^6} \Rightarrow P = 105\,\text{kN}$$

Check Shear

$$\tau_{xy,\text{allow}} = \frac{VQ}{Ib} = 0.73$$



$$Q = Ad$$

$$Q = \frac{bh}{2} \cdot \frac{h}{4} = \frac{bh^2}{8}$$

$$Q = 1750 \times 10^3\,\text{mm}^4$$

$$0.733 = \frac{VQ}{Ib}$$

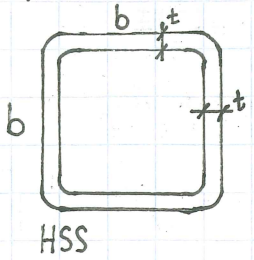
$$0.733 = \frac{P \cdot 1750 \times 10^3}{2 \cdot 281 \times 10^6 \cdot 241} \Rightarrow P = 56.7\,\text{kN}$$

∴ Member limited in safe load by shear to P = 56.7kN

CIV102 - STRUCTURES and MATERIALS

Topic: Thin Plate Buckling

1) Thin Wall Members

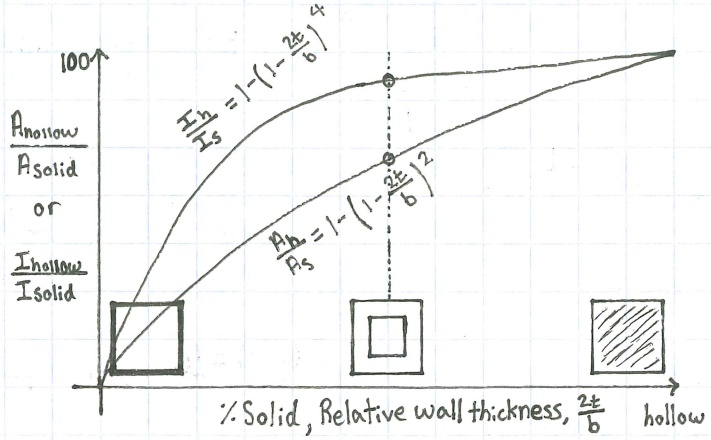


$$\text{Area} = b^2 - (b-2t)^2 \qquad I = \frac{b^4}{12} - \frac{(b-2t)^4}{12}$$

$$\frac{A_{hollow}}{A_{solid}} = \frac{b^2-(b-2t)^2}{b^2} = 1-\left(1-\frac{2t}{b}\right)^2 \quad \text{(weight related to this)}$$

$$\frac{I_{hollow}}{I_{solid}} = \frac{\frac{b^2}{12} - \frac{(b-2t)^4}{12}}{\frac{b^4}{12}} = 1-\left(1-\frac{2t}{b}\right)^4 \quad \text{(Strength related to this)}$$

Strength = quartic



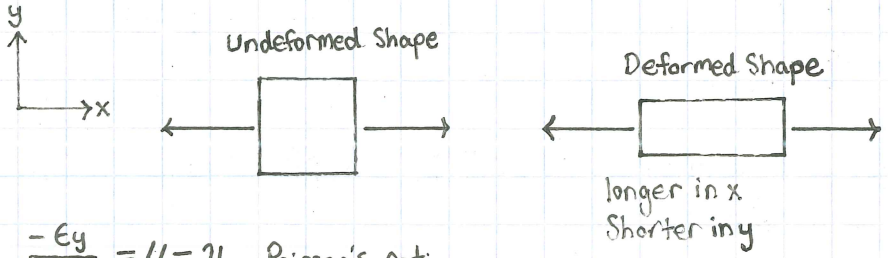
$t = \frac{b}{2} \longrightarrow$ Empty

Thinner walled members have better strength to weight ratios

2) Plate Buckling

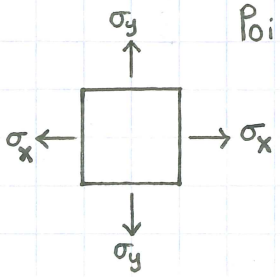
- Will limit how thin the wall can be
- Derivation is hard

2a) Poisson's Ratio



$$\frac{-\epsilon_y}{\epsilon_x} = \mu = \nu$$ Poisson's ratio is constant

rubber, $\mu = 0.5$
steel, $\mu = 0.3$
Cork, $\mu = 0$



Strain from stress adds to strain from Poisson's Ratio

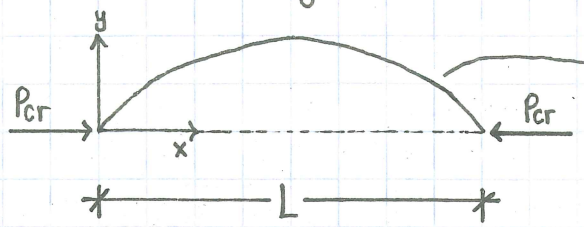
$$\epsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} \qquad \epsilon_y = \frac{-\mu\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$\sigma_x = \frac{E}{1-\mu^2}(\epsilon_x + \mu\epsilon_y)$$

$$\sigma_y = \boxed{\frac{E}{1-\mu^2}}(\mu\epsilon_x + \epsilon_y)$$

↳ Effective E?

2b) Euler Buckling



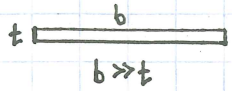
$$y(x) = A\sin\left(\frac{n\pi x}{L}\right)$$ (n: integer)

$$P_{cr} = \frac{\pi^2 E I}{L^2}$$

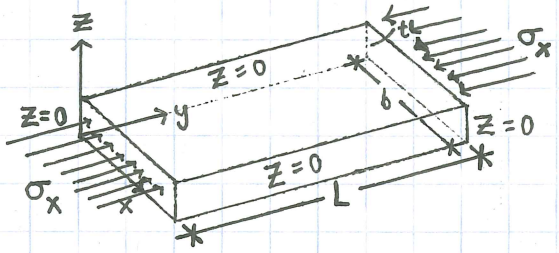
$$\frac{P_{cr}}{A} = \frac{\pi^2 E b t^3}{12 L^2 b t}$$

$$\sigma_{crit} = \frac{\pi^2 E}{12}\left(\frac{t}{L}\right)^2$$ Global Buckling/ Euler Critical Stress

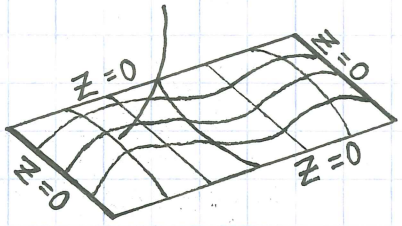
Cross Section



2c) Local Buckling Equation



Buckles Between Boundaries

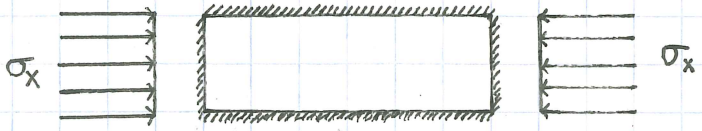


$$z(x,y) = A\sin\left(\frac{m x \pi}{L}\right)\sin\left(\frac{n y \pi}{L}\right)$$ m,n are 1,2,3,...

3) Thin Plate Buckling Equation

$$\sigma_{cr} = \frac{K\pi^2 E}{12(1-\mu^2)}\left(\frac{t}{b}\right)^2$$ if $L < b$ use $\left(\frac{t}{L}\right)^2$

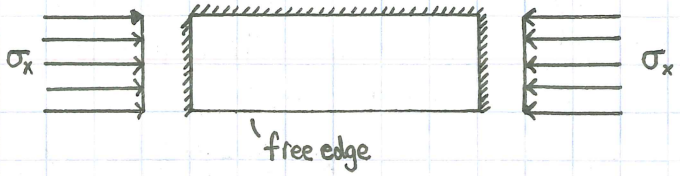
Case 1 – Restrained Against z Displacements on all 4 Sides

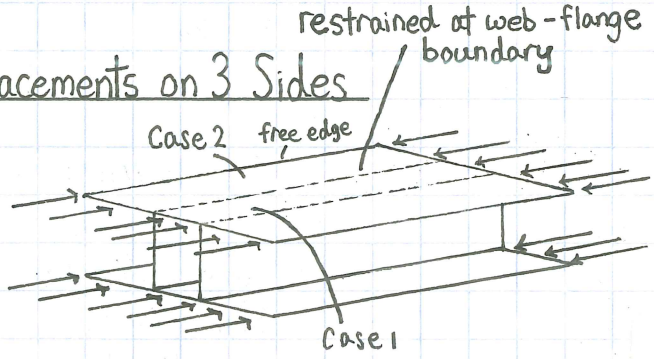


K = 4

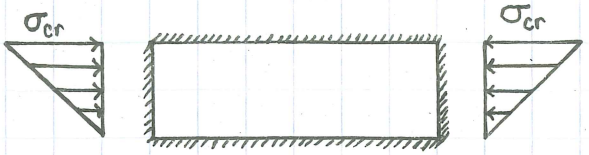
Case 2 – Restrained Against z Displacements on 3 Sides

K = 0.425



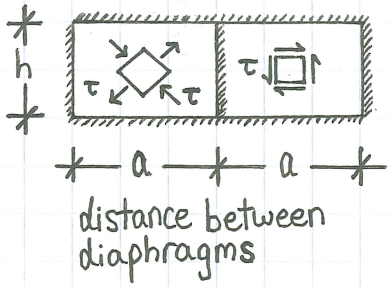


Case 3



$K = 6$

Case 4 – Shear Buckling

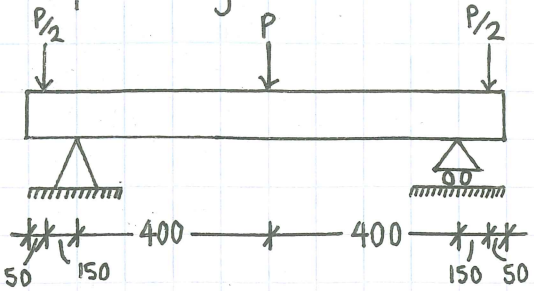


$$\tau_{cr} = \frac{5\pi^2 E}{12(1-\mu^2)}\left[\left(\frac{t}{h}\right)^2 + \left(\frac{t}{a}\right)^2\right]$$

CIV102 - STRUCTURES and MATERIALS

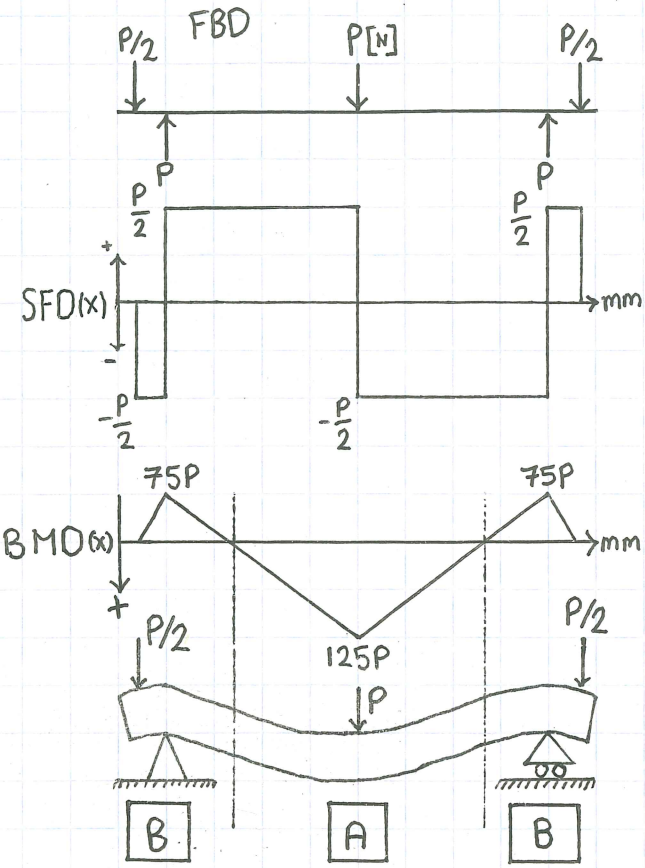
Topic: Example Bridge Design Part I

1) Example Bridge



$\mu = 0.2$
$E = 4000\,MPa$
$\sigma_{Tension} = 30\,MPa$
$\sigma_{compression} = 6\,MPa$

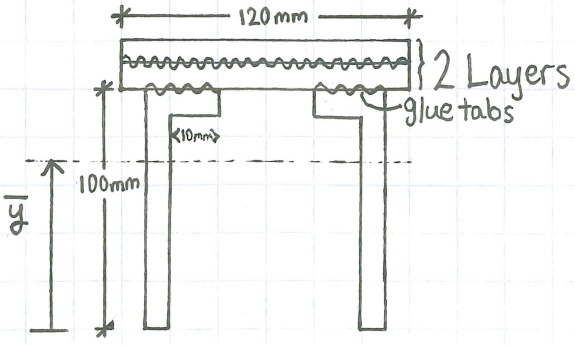
$\tau_{max} = 4\,MPa$
$\tau_{glue} = 2\,MPa$
$t = 1.27\,mm$



Displaced Shape

Section A = Tension on Bottom
Section B = Tension on Top

Section A



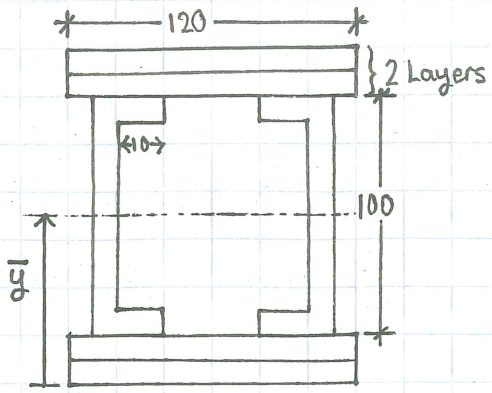
$\bar{y} = 78.9\,mm$
$y_{Bot} = 78.9\,mm$

$y_{Top} = 23.6\,mm$
$I = 587 \times 10^3\,mm^4$

$\frac{y_{Bot}}{y_{Top}} = \frac{78.9}{23.6} \approx 3$

Best $\frac{\sigma_{Ten}}{\sigma_{comp}} = 5 = \frac{30}{6}$

Section B



Check Flexural Strength

Section A

Compression	Tension
$\sigma = \frac{My}{I}$	$\sigma = \frac{My}{I}$
$6 = \frac{125P_1 \cdot 23.6}{587 \times 10^3}$	$30 = \frac{125P_2 \cdot 78.9}{587 \times 10^3}$
$P_1 = 1193N$	$P_2 = 1786N$

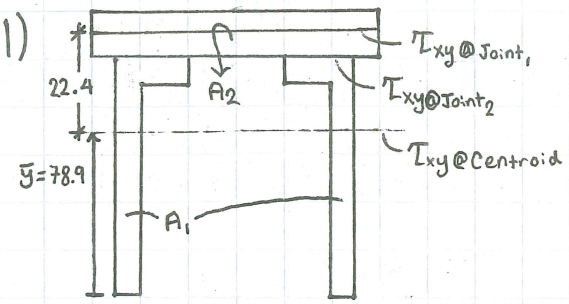
Section B

Compression	Tension
$\sigma = \frac{My}{I}$	$\sigma = \frac{My}{I}$
$6 = \frac{75P_3 \cdot 52.5}{1938 \times 10^3}$	$30 = \frac{75P_4 \cdot 52.5}{1938 \times 10^3}$
$P_3 = 2950N$	$P_4 = 14754N$

CIV102 - STRUCTURES and MATERIALS

Topic: Example Bridge Design Part II

1)



Shear at Glue Joint$_2$

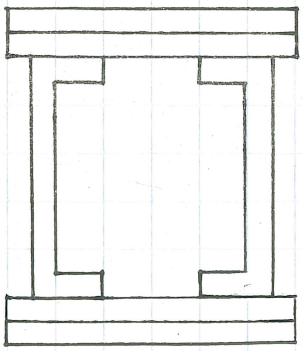
$$V = \frac{\tau_{xy} I \cdot b}{Q}$$

$$Q_2 = A_2 \cdot dist = 1.27 \times 2 \times 120 \times (22.4)$$
$$Q_2 = 6806 mm^3$$

$$\frac{P_6}{2} = \frac{\tau_{xy} \cdot I \cdot b}{Q_2} \Rightarrow \frac{P_6}{2} = \frac{2 \times 587 \times 10^3 \times (2 \times 10)}{6806}$$

$$P_6 = 6899N$$

Shear at [B]



$$\frac{I}{Q} = \text{Bigger}$$
$$\frac{I}{Q} = \text{Bigger}$$

$$\tau_{xy} = \frac{VQ}{Ib}$$

Shear

[A] Max τ_{xy} = 4MPa

$$\tau_{xy} = \frac{VQ}{Ib} \Rightarrow V = \frac{\tau_{xy} I \cdot b}{Q}$$

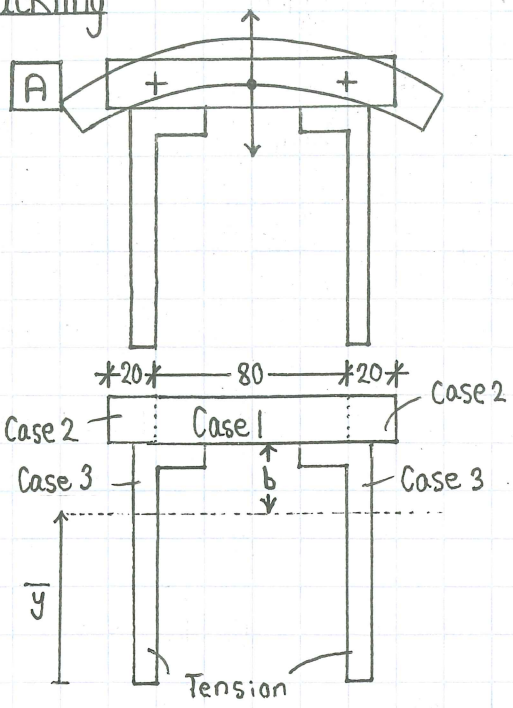
$$Q_1 = A_1 \cdot dist = 78.9 \times 1.27 \times 2 \times \frac{78.9}{2}$$
$$Q_1 = 7906 mm^3$$

$$\frac{P_5}{2} = \frac{\tau_{xy} \cdot I \cdot b}{Q_1} \Rightarrow \frac{P_5}{2} = \frac{4 \cdot 587 \times 10^3 (1.27 \cdot 2)}{7906}$$

$$P_5 = 1509N$$

Shear at Glue Joint$_1$

Buckling



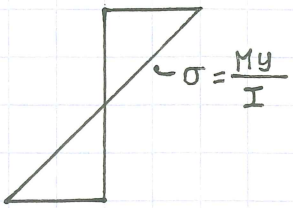
Case 1 - Restrained on 2 sides, Constant σ

Case 2 - Restrained on 1 side, Constant σ

Case 3 - Restrained on 2 sides, variable σ

Case 4 - Shear Stress

Axial Stress → Moments



Case 1 Buckling Top Centre

$$\sigma_{cr} = \frac{K \pi^2 E}{12(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

$$= \frac{4 \pi^2 4000}{12(1-0.2^2)} \left(\frac{1.27 \cdot 2}{80}\right)^2$$

$$= 13.8 \text{ MPa}$$

Since > 6 MPa ignore this buckling

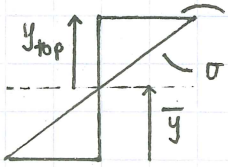
Case 2

$$\sigma_{cr} = \frac{0.425 \pi^2 E}{12(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

$t = 1.27 \times 2 \quad b = 20$

$\sigma_{cr} = 23.5 \text{ MPa}$

How to Respond if $\sigma_{cr} < 6$ MPa ?



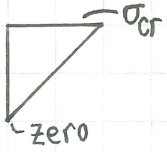
Make this $\sigma = \sigma_{cr}$ Solve for M, Solve for P

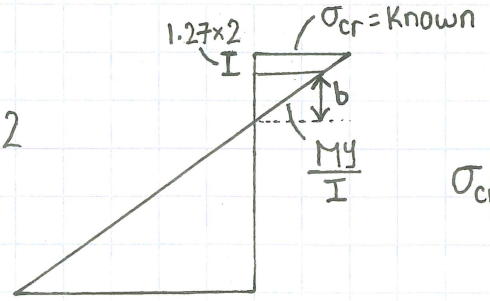
Case 3

$b = 23.6 - 1.27 \times 2$

$t = 1.27 \text{ mm}$

$K = 6$



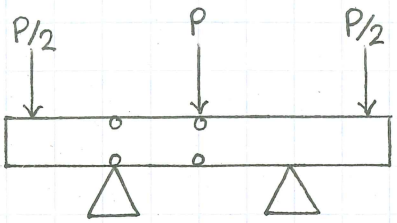


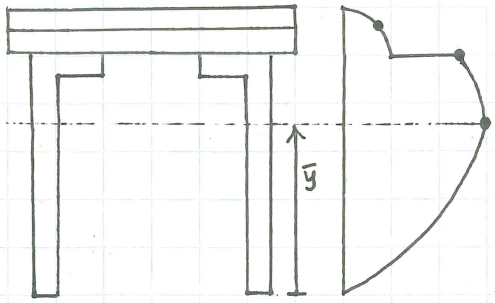
$$\sigma_{cr} = \frac{M(236 - 1.27 \times 2)}{I}$$

CIV102 - STRUCTURES and MATERIALS

Topic: Example Bridge Design Part III

1)





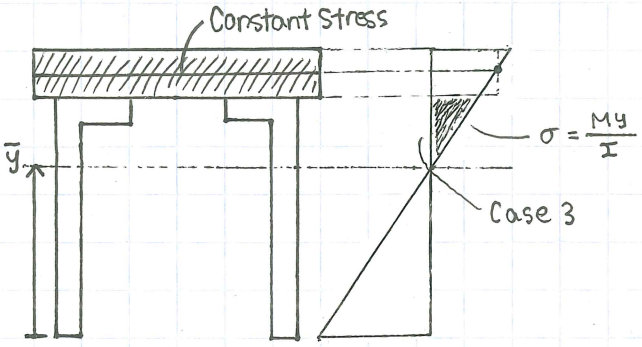
SFD/BMD → Failure By $\begin{cases} \sigma_{ten} \\ \sigma_{comp} \end{cases}$

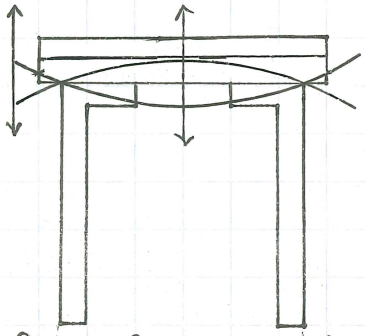
SFD → Failure By $\begin{cases} \tau_{xy\,max} \\ \tau_{xy\,glue} \end{cases}$

Buckling

Case 1
Case 2
Case 3
] $\sigma - M$

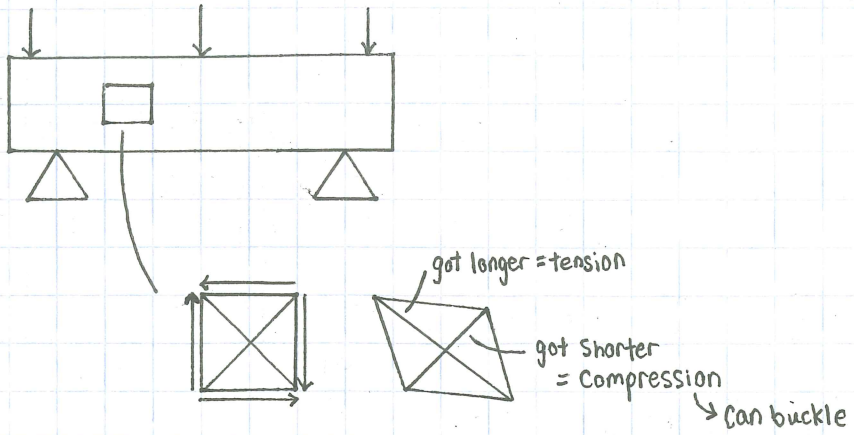
Case 4 → τ_{xy}

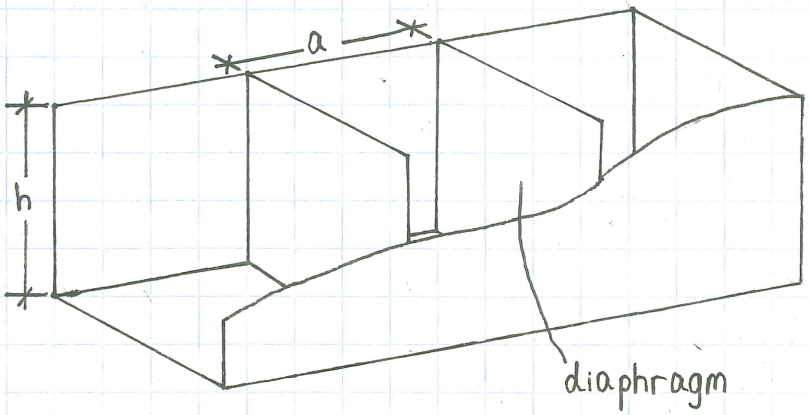




Case 2 | Case 1 | Case 2

2) Shear Buckling





$$\tau_{xy} = \frac{5\pi^2 E}{12(1-\mu^2)}\left[\left(\frac{t}{a}\right)^2 + \left(\frac{t}{h}\right)^2\right]$$

τ_{crit} = buckling shear stress

$$= \frac{5\pi^2 4000}{12(1-0.2^2)}\left[\left(\frac{1.27}{200}\right)^2 + \left(\frac{1.27}{100}\right)^2\right]$$

$\tau_{crit} = 3.45\,MPa$

↑ new failure stress, recalc P by $\frac{VQ}{IB}$

CIV102 - STRUCTURES and MATERIALS

Topic: Stone and Masonry

1) Stone

- good in compression
- fairly weak in tension
- at joints no tensile strength
- You can't pull on a stone

2T3 student said → Stone is anti-rope

Advantages

- Material is cheap
  - ↳ but cutting stones = expensive
- last long time

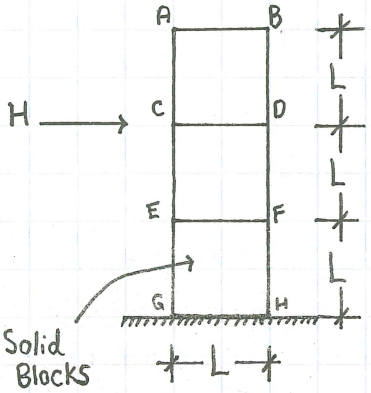
How do we design?

Allowable Stress / Tensile = Zero MPa
Allowable Stress \ Compressive = $\frac{\text{Strength}}{\text{FOS}}$

Shear is often ignored

Design Methods

1) Combined Stresses

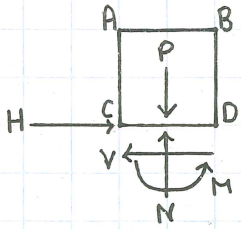


Each block weighs P kN

Joints Will be Weaker Spots

Look at Joint C-D

F.B.D.

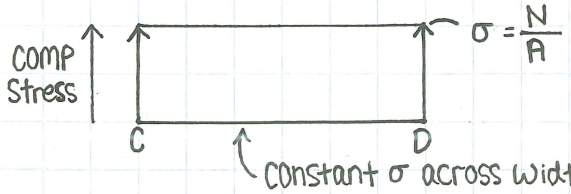


$\Sigma F_x = 0$

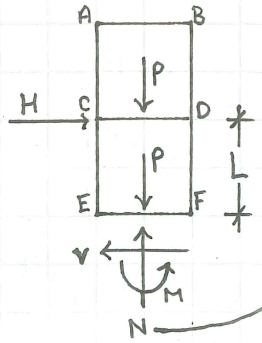
$V = H$

$\Sigma M = 0$

$N = P$



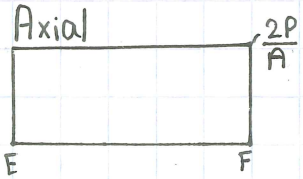
EF

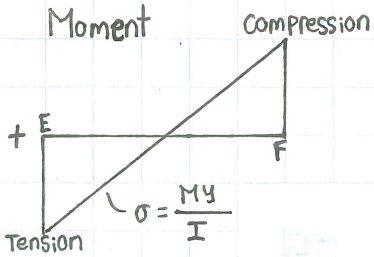


$V = H$

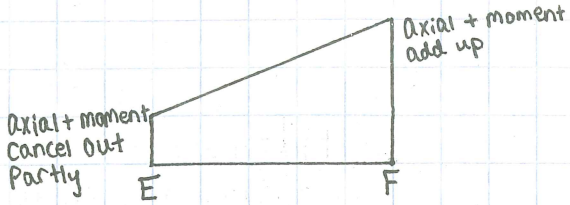
$N = 2P$

$M = HL$

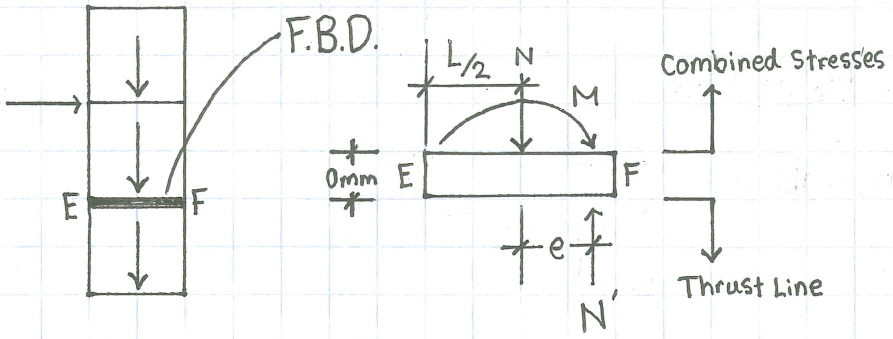




Take Sum of Axial + Moment Stress



2) Thrust Line



$\Sigma F_y = 0$

$0 = -N + N' \Rightarrow N = N'$

$\Sigma M_{centroid} = 0 = -m + Ne$

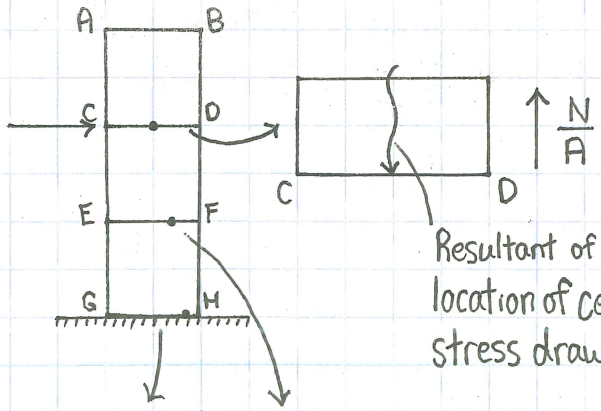
$\Rightarrow e = \frac{M}{N}$

e = eccentricity from thrust line to centroid

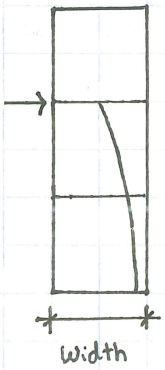
For combined σ (M+N) take sum of stresses and compare to allowable

$$\sigma_{Total} = \frac{N}{A} + \frac{My}{I}$$

Thrust line has only an axial force not necessarily at centroid of X-section

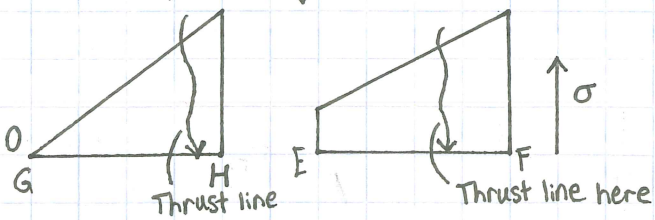


Trace the Thrust Line

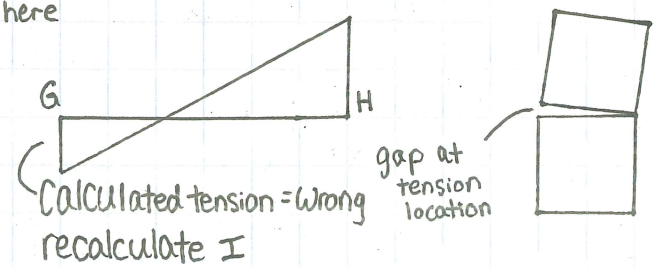


If thrust line is in middle 1/3 of width → no tensile stress = Kern

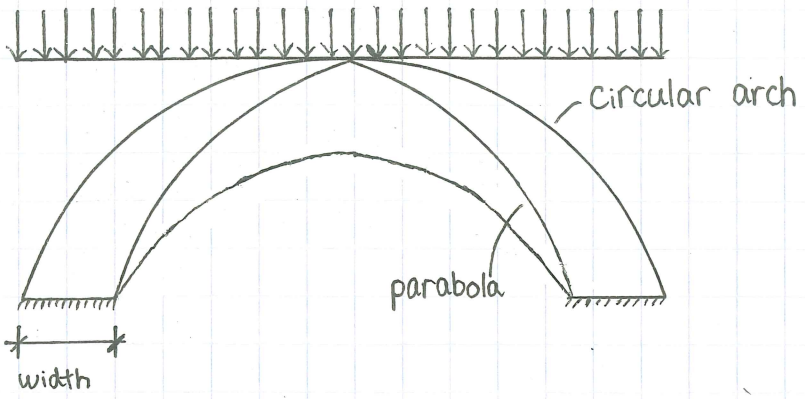
If thrust line goes outside width it will collapse



If thrust line is closer than 1/3 from edge



If shape matches thrust line I get an arch



What shape is best for an arch?

Robert Hooke 1675

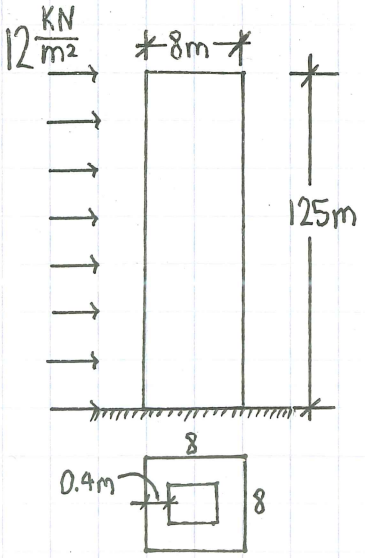
"As hangs the flexible cable, so inverted should the touching parts of an arch stand."

Allowable compressive stress for stone or brick = $\frac{\sigma_{compressive}}{F.O.S.}$ ← from table

$\sigma_{Tensile\ Allowable}$ = zero!

Example

Chimney at St. Rollox

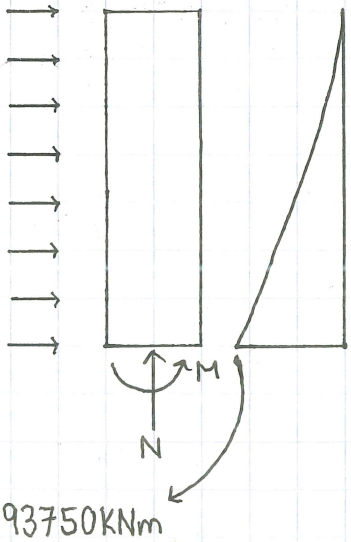


$I = 117.4 m^4$

$A = 12.1 m^2$

$\gamma = 26 \frac{KN}{m^3}$

FBD



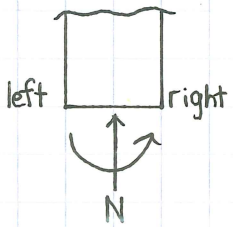
$M = 93750 KNm$

$N = A\gamma h = 12.1 \cdot 26 \cdot 125$

$= 39520 KN$

M_{max}, N_{max} @ Bottom

∴ Calculate there



$\frac{N}{A} = 3.25 MPa$ compression

$\sigma_{left} = 3.25 - \frac{My}{I}$

$= 0.06 MPa$ (comp)

$\sigma_{right} = 3.25 + \frac{My}{I}$

$= 6.44 MPa$

CIVI02 - STRUCTURES and MATERIALS

Topic: Reinforced Concrete

1) Concrete

- Synthetic Stone
- Made from:
  - Cement
  - Sand] Volume instead of cement
  - Aggregate (Gravel)] Volume stability (does not shrink)
  - Water] Water + Cement turns into solid
  - Air] Water expands when freezes and can break
  - Admixtures concrete, air is needed to stop it

Cement and concrete are not the same
↳ Cement is more powdery, used to make concrete
(Cement : Concrete → flour : Cake)

When concrete sets, it gets smaller

Cement

- Made of grinded up limestone
- Heated up to over 1000°C with coal
  ↳ Gives off CO_2 and water
- Turns into ceramic
- Grind up the ceramic into cement
  ↳ add water → lime stone

Why Concrete?

- Most used construction material
- Cheap to make and export
- Local production
- Durable
- Make any shape (Since it starts as a liquid)

Negative: Carbon intensive

2) Types of Concrete

a) Plain Concrete

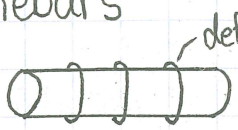
- No reinforcing steel
- Like stone
  ↳ But has time dependent properties
- Carries compression well, but not tension

b) Reinforced Concrete

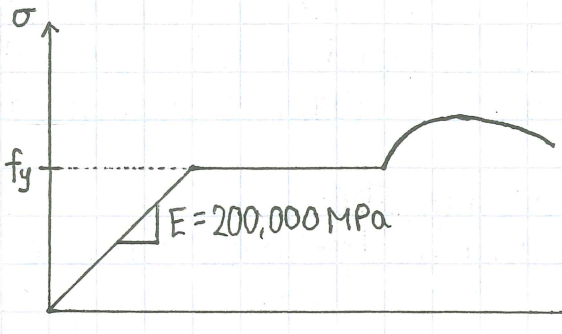
- Concrete + Steel Reinforcing Bars (Rebar) where there is tension
- Use deformed rebars

C) Prestressed Concrete
- Reinforced concrete where before loading, steel is in tension and concrete is in compression

3) Rebars



Helps to bond the steel and concrete
Rusty bars have a better bond



Ignore Strain Hardening

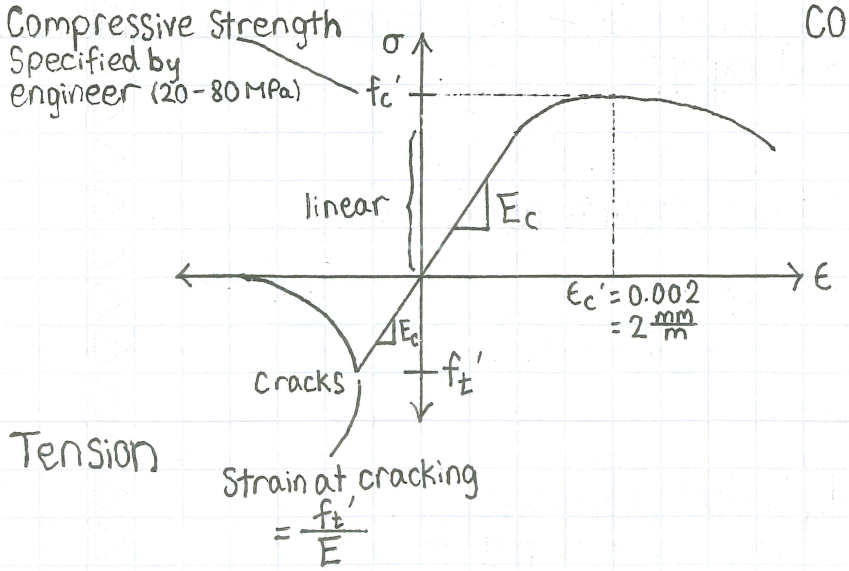
f_y = 400MPa
Rest of World, f_y = 500MPa

Standard Bar Sizes

Size	Area
10M	100mm^2
15M	200mm^2
20M	300mm^2
25M	500mm^2
30M	700mm^2
35M	1000mm^2

The size relates to its diameter

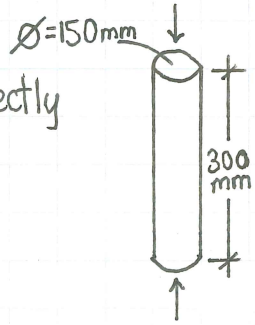
4) Concrete Stress - Strain Relationship (not reinforced)



compression

Must Consider f_c', f_t', and E when working with concrete.

f_c' → Measured in compression test @ 28 days of age (test all trucks)



f_t' → Estimated indirectly
$f_t' = 0.33\sqrt{f_c'}$

$E_c = 4730\sqrt{f_c'}$

Composite Material

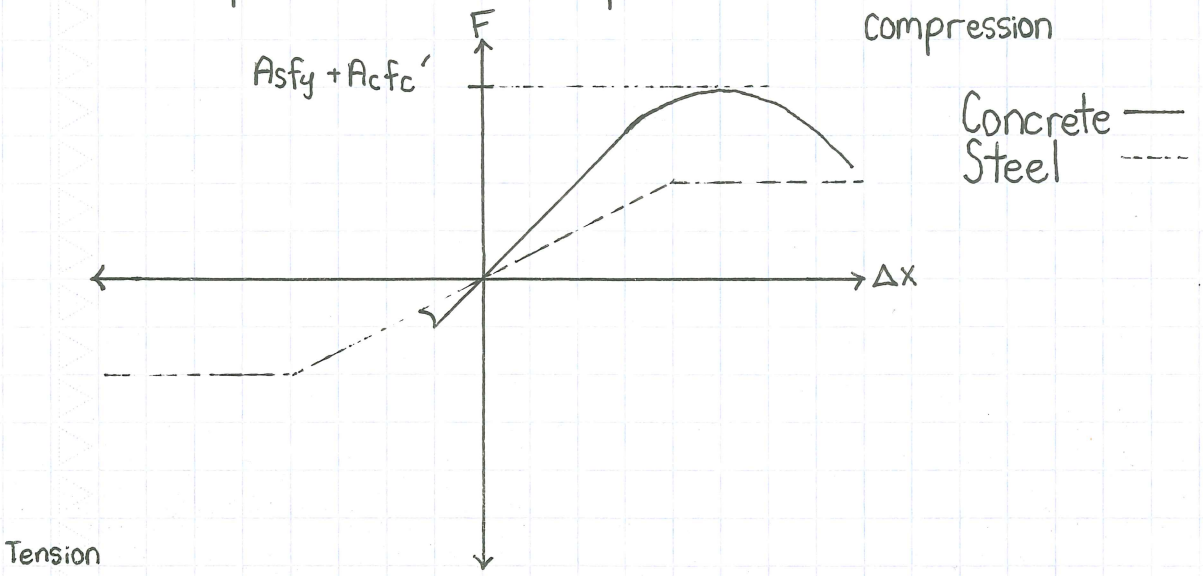
- Contains at least 2 different materials
- Assume that concrete and steel have same strain

ϵ = Constant f_c = concrete stress ($\epsilon \cdot E_c$)
f_s = Steel Stress ($\epsilon \cdot E_s$)

Max Compressive Capacity $\simeq f_c' \cdot A_c + f_y \cdot A_s$

- A_c = area of concrete (total area - A_s)
- A_s = area of steel rebar
- Able to do this since yield (f_y) and crush (f_c') happens at the same strain

Graph of Force vs Displacement

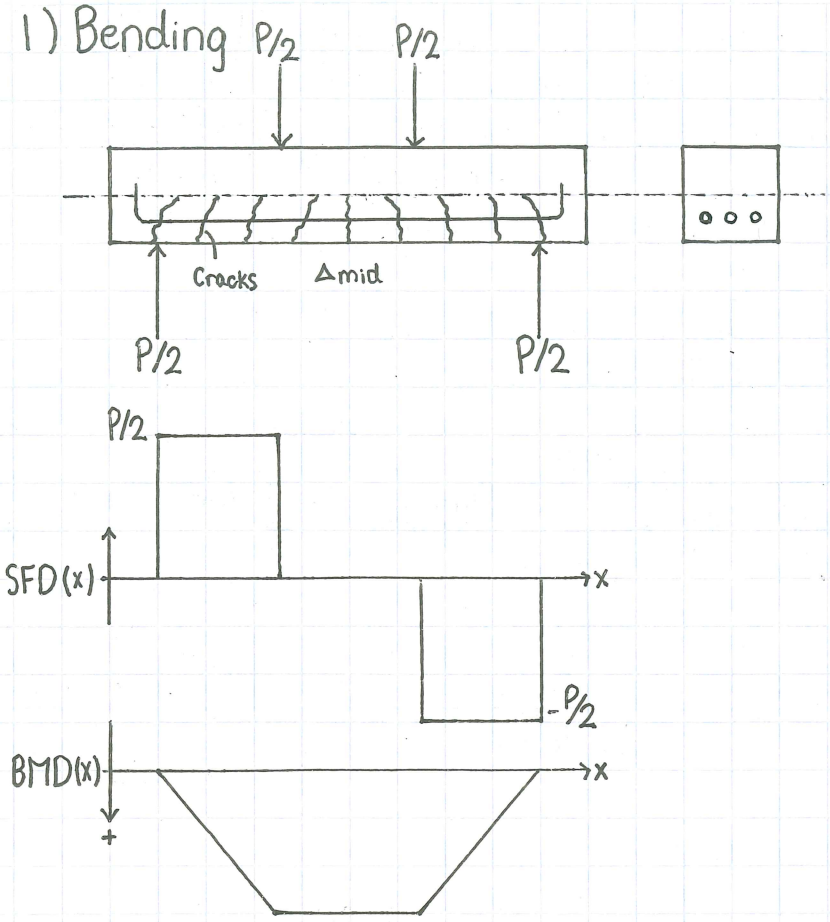


- After cracking, the concrete does not do anything

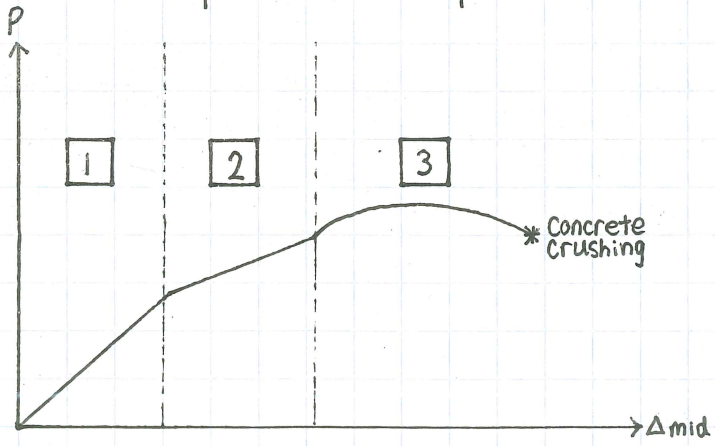
CIV102 - STRUCTURES and MATERIALS

Topic: Reinforced Concrete Bending

1) Bending



Force vs Displacement Graph of Reinforced Concrete

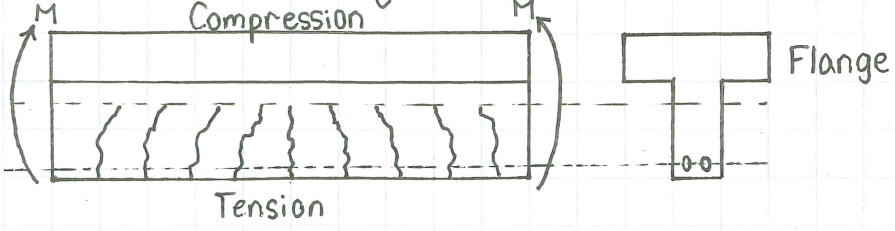


1 Uncracked Elastic

2 Cracked Elastic

3 Nonlinear Yielding

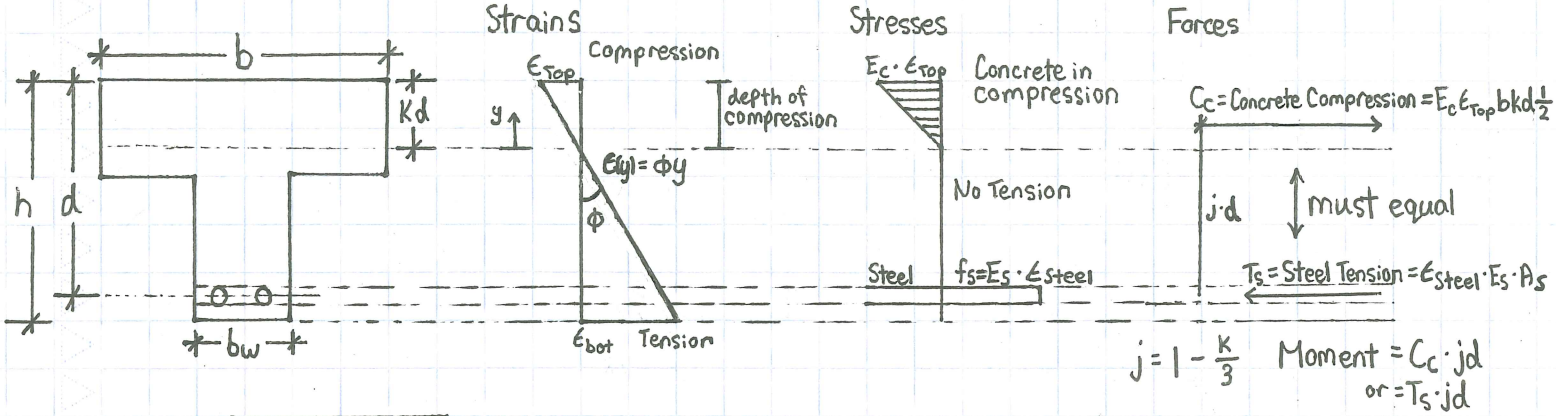
Look at Member Subjected to M alone in 2



Assumptions

- Hooke's Law Applies to < Concrete in compression / Steel in tension
- No tensile concrete stress at all (Axial load = Zero)
- Plane sections remain plane
- Bond between steel and concrete is perfect
  - ↳ ∴ $\epsilon_{concrete} = \epsilon_{steel}$ at depth of rebar
- Depth of compression is at flange

FOS for concrete in compression = 2
FOS for steel in tension = 1.67



$$K = \sqrt{(n\cdot\rho)^2 + 2n\rho} - n\rho$$

n = modular ratio = $\frac{E_s}{E_c}$

ρ = reinforcement percentage = $\frac{A_s}{bd}$

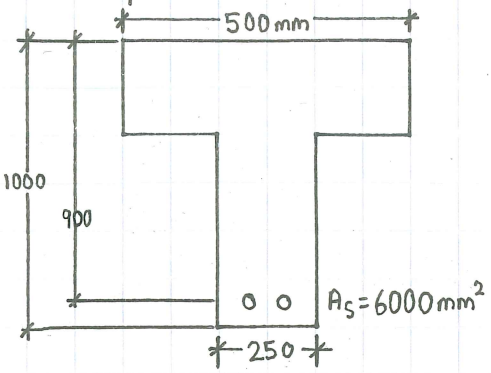
$$M = T_s \cdot j \cdot d$$
$$= A_s f_s \cdot j \cdot d \Rightarrow \underline{\underline{f_s = \frac{M}{A_s \cdot j \cdot d} \leq \frac{f_y}{FOS}}}$$

$$f_s = E_c \cdot \epsilon_{Top}$$

$$\epsilon_{Top} = \frac{K}{1-K} \cdot \epsilon_{steel} \qquad \epsilon_{steel} = \frac{f_s}{\epsilon_s}$$

$$\underline{\underline{f_c = \frac{K}{1-K} \frac{M}{j \cdot d \cdot A_s \cdot n} \leq \frac{f_c'}{FOS}}}$$

Example



$f_c' = 40 MPa$
$f_y = 400 MPa$

Moment = 1200KNm Is this safe?

1. $E_c = 4500\sqrt{f_c'}$
$= 28500 MPa$

2. $n = \frac{E_s}{E_c} = \frac{200000}{28500}$
$= 7.03$

3. $\rho = \frac{A_s}{b \cdot d} = \frac{6000}{500 \cdot 900}$
$= 0.01333$
or
1.3%

4. $k = \sqrt{(n\rho)^2 + 2n\rho} - n\rho$
$= 0.349$

5. $j = 1 - \frac{k}{3}$
$= 0.884$

Check f_s

$f_s = \frac{M}{A_s \cdot j \cdot d} = \frac{1200 \times 10^6}{6000 \cdot 0.884 \cdot 900}$
$= 251 MPa$

Allowable $= \frac{f_y}{1.67} = 239 \rightarrow$ Not Safe

Check f_c

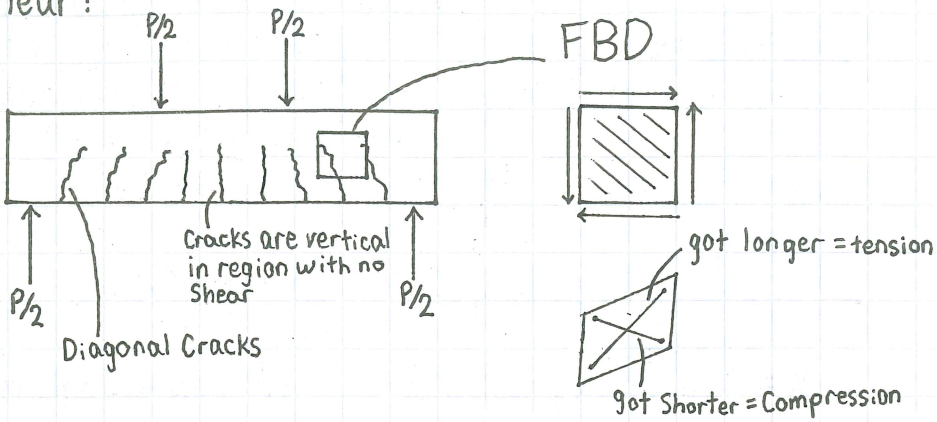
$f_c = \frac{k}{(1-k)} \frac{M}{j \cdot d \cdot A_s \cdot n} = 19.17 MPa$

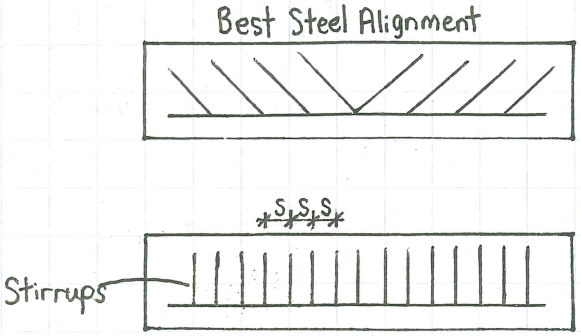
Allowable $= \frac{f_c'}{FOS} = \frac{40}{2} = 20 MPa \rightarrow$ ok for f_c

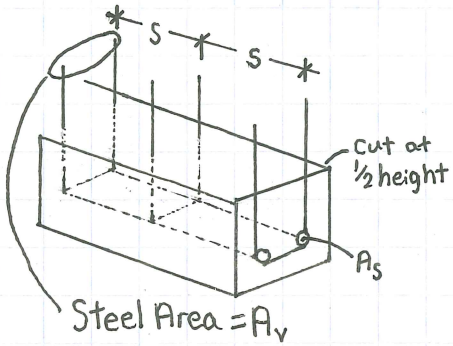
CIV102 - STRUCTURES and MATERIALS

Topic: Reinforced Concrete Design for Shear

1) Shear!

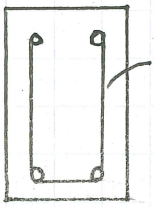






Steel Area = A_v

= (Area of Bar)(# of legs across width)

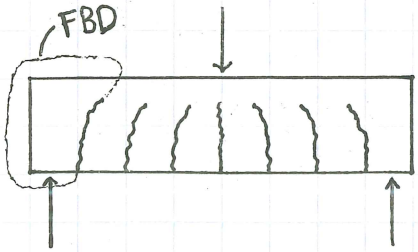


2 legs of stirrups

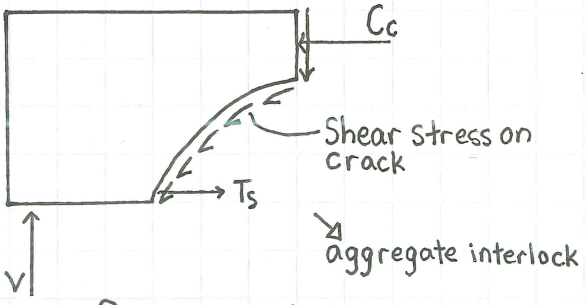
- if 10M = 100mm²

$A_v = 200mm^2$

2) How is Shear carried?

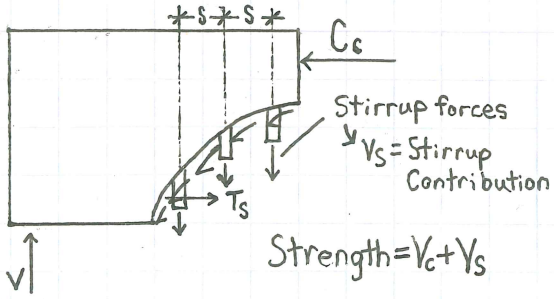


a) If no stirrups



- Concrete contribution to shear Strength V_c

b) With Shear Reinforcement



Strength = $V_c + V_s$

V_c has no size effect if you have stirrups

V_c shows a size effect. Members with larger depth, d or h, show lower shear stress @ failure

3) Design for Shear

$V_{ultimate} = V_c + V_s \leq V_{max}$

$V_{safe} = \frac{V_c}{FOS} + \frac{V_s}{FOS} \leq \frac{V_{max}}{FOS}$ → To avoid concrete crushing in diagonal compression

↳ 2.0 ↳ 1.67 ↳ 2.0

$V_{max} = 0.25 f_c' \cdot b_w \cdot j \cdot d$

If we have no stirrups

$V_c = \frac{230\sqrt{f_c'}}{1000 + 0.9d} b_w \cdot j \cdot d$

$V_s = 0$

If we have more than minimum stirrups

We have enough if $\frac{A_v \cdot f_y}{b_w \cdot s} \geq 0.06\sqrt{f_c'}$

$V_c = 0.18\sqrt{f_c'}\, b_w \cdot j \cdot d$

$V_s = \frac{A_v \cdot f_y \cdot j \cdot d}{s} \cot\theta \qquad \theta = 35°$

CIV102 - STRUCTURES and MATERIALS

Topic: Shear Design + Example

1) Shear Design

Shear Force → SFD
- ↳ Use max |Shear| value
- ↳ $V_{applied}$

$V_{applied} \leq V_{safe}$

$V_{applied} \leq \frac{V_c}{FOS} + \frac{V_s}{FOS} \leq \frac{V_{max}}{FOS}$ → Crushing of concrete

(V_s: yielding of stirrups; $\frac{V_c}{FOS}$: Sliding on concrete crack)

- $\frac{V_c}{FOS}$: ↳ 2.0
- $\frac{V_s}{FOS}$: ↳ 1.67
- $\frac{V_{max}}{FOS}$: ↳ 2.0

How to design for shear in RC

i) Need $V_{applied}$ ← SFD

ii) Given info f_c', f_y, b_w, b, shape of cross section, A_s

iii) Check $V_{max} = 0.25 f_c' \cdot b_w \cdot j \cdot d$ if $V_{applied} < \frac{V_{max}}{2.0}$ ok

iv) Assume we do not need shear reinforcement

$$V_c = \frac{230\sqrt{f_c'}}{1000 + 0.9d} b_w \cdot j \cdot d$$ if $V_{applied} < \frac{V_c}{2.0}$ → ok

if not, add stirrups

v) You do need shear reinforcement

$V_c = 0.18\sqrt{f_c'} \cdot b_w \cdot j \cdot d$ (this will only be correct if you have at least a minimum quantity of stirrups)

Check spacing (S_1) to get minimum shear steel

$$\frac{A_v \cdot f_y}{b_w \cdot S_1} \geq 0.06\sqrt{f_c'}$$ (Solve for S_1 to exactly match the minimum)

Also check that resulting meets the general limit

$$S_2 \leq 0.7 j \cdot d$$

Take $\min\{S_1, S_2\} = S$

$$V_s = \frac{A_v \cdot f_y \cdot j \cdot d}{S} \cot\theta \quad \theta = 35°$$

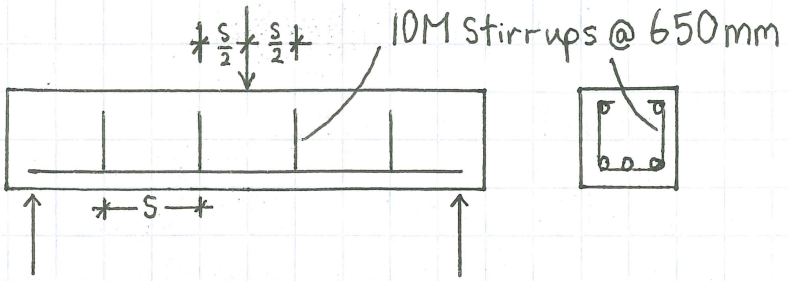
Check $V_{applied} \leq \frac{V_c}{2.0} + \frac{V_s}{1.67}$ if yes → done
if no → Select new, smaller S

Calculate required value of S

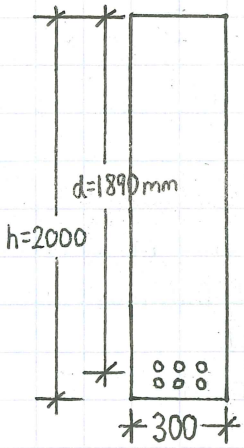
$$V_{applies} = \frac{V_c}{2.0} + \frac{1}{1.67} \cdot \frac{A_v \cdot f_y \cdot j \cdot d}{S} \cdot \cot 35°$$

Solve for S. Round down value of S to nearest say 25mm.

vi) Draw a sketch



2) Example 1



$f_c' = 35.4MPa$
$f_y = 450MPa$

$j \cdot d = 1715mm$ from flexural safety check of f_c, f_s

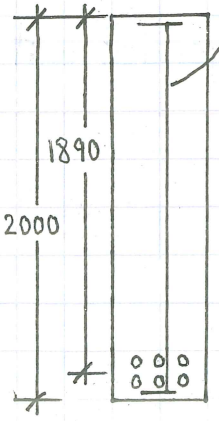
What is predicted failure shear

$V_{max} = 0.25 \cdot f_c' \cdot b_w \cdot j \cdot d = 4553kN$

$V_c = \frac{230\sqrt{35.4}}{1000 + 0.9 \cdot 1890} \cdot 300 \cdot 1715 = 261kN < V_{max}$ estimate

Test Strength = 256kN ok

3) Example 2



Area of a US#4 = 129mm²
∵ Only one leg in beam
$A_v = A_{i\,bar} = 129mm^2$

f_y - Stirrup = 468MPa

$b_w = 300$
$f_c' = 36.6$

Do we have min shear reinforcement?

$\frac{A_v \cdot f_y}{b_w \cdot S} \geq 0.06\sqrt{f_c'}$ $V_c + V_s = 811kN < V_{max} = 4553kN$

$0.341 \geq 0.363$ $V_{test} = 674kN$
for us, close enough ↳ 84% of predicted

$V_c = 0.18\sqrt{f_c'} b_w \cdot j \cdot d = 560kN$

$V_s = \frac{A_v \cdot f_y \cdot j \cdot d}{S} \cot 35 = 251kN$